3.3 Proving and Applying the Cosine Law

YOU WILL NEED

- ruler
- protractor
- calculator

GOAL

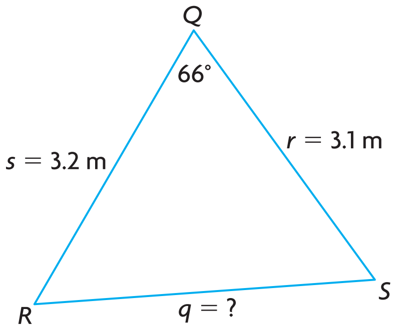
Explain the steps used to prove the cosine law. Use the cosine law to solve triangles.

EXPLORE…

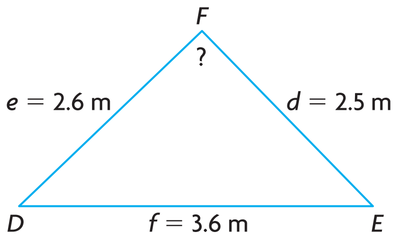
- One side of a right triangle is 8 cm. One angle is 50°. What could the other side lengths be?

INVESTIGATE the Math

The sine law cannot always help you determine unknown angle measures or side lengths. Consider these triangles:



where $\dfrac{3.1}{\sin R} = \dfrac{3.2}{\sin S} = \dfrac{q}{\sin 66°}$



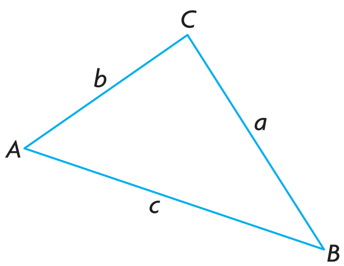
where $\dfrac{\sin E}{2.6} = \dfrac{\sin D}{2.5} = \dfrac{\sin F}{3.6}$

There are two unknowns in each pair of equivalent ratios, so the pairs cannot be used to solve for the unknowns. Another relationship is needed. This relationship is called the **cosine law**, and it is derived from the Pythagorean theorem.

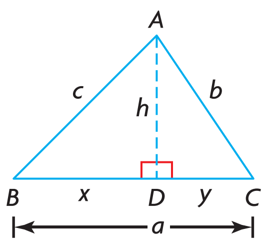
cosine law

In any acute triangle,

$a^2 = b^2 + c^2 - 2bc \cos A$
$b^2 = a^2 + c^2 - 2ac \cos B$
$c^2 = a^2 + b^2 - 2ab \cos C$



Before this relationship can be used to solve problems, it must be proven to work in all acute triangles. Consider Heather's proof of the cosine law:



Step 1
I drew an acute triangle *ABC*. Then I drew the height from *A* to *BC* and labelled the intersection point as point *D*. I labelled this line segment *h*. I labelled *BD* as *x* and *DC* as *y*.

$$h^2 = c^2 - x^2$$
$$h^2 = b^2 - y^2$$

Step 2
I wrote two different expressions for h^2.

$$c^2 - x^2 = b^2 - y^2$$
$$c^2 = x^2 + b^2 - y^2$$

Step 3
I set the two expressions equal to each other and solved for c^2.

$x = a - y$, so
$$c^2 = (a - y)^2 + b^2 - y^2$$
$$c^2 = a^2 - 2ay + y^2 + b^2 - y^2$$
$$c^2 = a^2 + b^2 - 2ay$$

Step 4
I wrote an equivalent equation that only used the variable y and simplified.

$\cos C = \dfrac{y}{b}$, so
$$b \cos C = y$$

Step 5
I determined an equivalent expression for y.

$$c^2 = a^2 + b^2 - 2ay$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Step 6
I substituted the expression $b \cos C$ for y in my equation.

? How can you improve Heather's explanations in her proof of the cosine law?

A. Work with a partner to explain why she drew height AD in step 1.

B. In step 2, Heather created two different expressions that involved h^2. Explain how she did this.

C. Explain why she was able to set the expressions for h^2 equal in step 3.

D. In step 4, Heather eliminated the variable x. Explain how and why.

E. Explain how she determined an equivalent expression for y in step 5.

F. Explain why the final equation in step 6 is the most useful form of the cosine law.

Reflecting

G. Les wrote a similar proof, but he substituted $a - x$ for y instead of $a - y$ for x in the equation in step 3. How would his result differ from Heather's?

H. François started his proof by drawing a height from B to AC. How would this affect his final result?

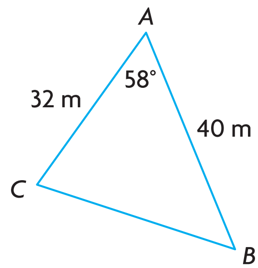
I. **i)** Explain why you can use the cosine law to determine the unknown side q in $\triangle QRS$ on the previous page.

ii) Explain why you can use the cosine law to determine the unknown $\angle F$ in $\triangle DEF$ on the previous page.

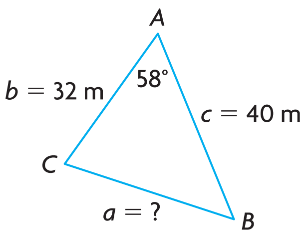
APPLY the Math

EXAMPLE 1 Using reasoning to determine the length of a side

Determine the length of CB to the nearest metre.



Justin's Solution



I labelled the sides with letters.
I couldn't use the sine law, because I didn't know a side length and the measure of its opposite angle.

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$a^2 = 32^2 + 40^2 - 2(32)(40) \cos 58°$$

I knew the lengths of two sides (b and c) and the measure of the contained angle between these sides ($\angle A$). I had to determine side a, which is opposite $\angle A$. I chose the form of the cosine law that includes these four values. Then I substituted the values I knew into the cosine law.

$$a^2 = 1024 + 1600 - 2560 \cos 58°$$
$$a^2 = 2624 - 2560 \cos 58°$$
$$a^2 = 1267.406...$$
$$a = \sqrt{1267.406...}$$
$$a = 35.600...$$

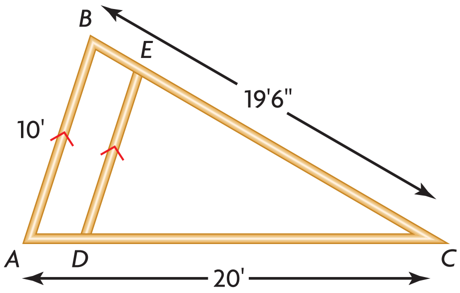
CB is 36 m.

Your Turn

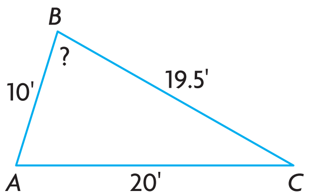
After determining the length of CB in $\triangle ABC$ above, Justin used the sine law to determine that the measure of $\angle B$ is 50°, then concluded that the measure of $\angle C$ must be 72°. Use the cosine law to verify his solution for $\triangle ABC$.

EXAMPLE 2 Using reasoning to determine the measure of an angle

The diagram at the right shows the plan for a roof, with support beam *DE* parallel to *AB*. The local building code requires the angle formed at the peak of a roof to fall within a range of 70° to 80° so that snow and ice will not build up. Will this plan pass the local building code?



Luanne's Solution: Substituting into the cosine law, then rearranging



I drew a sketch, removing the support beam since it isn't needed to solve this problem.

The peak of the roof is represented by $\angle B$.

I labelled the sides I knew in the triangle.

I wrote all the lengths using the same unit, feet.

$a = 19.5,\ b = 20$, and $c = 10$

Since I only knew the lengths of the sides in the triangle, I couldn't use the sine law.

$$b^2 = a^2 + c^2 - 2ac \cos B$$

I had to determine $\angle B$, so I decided to use the form of the cosine law that contained $\angle B$.

$$\begin{aligned} 20^2 &= 19.5^2 + 10^2 - 2(19.5)(10) \cos B \\ 400 - 380.25 - 100 &= -390 (\cos B) \\ -80.25 &= -390 (\cos B) \\ \frac{-80.25}{-390} &= \cos B \\ \cos^{-1}\left(\frac{80.25}{390}\right) &= \angle B \\ 78.125...° &= \angle B \end{aligned}$$

I substituted the side lengths into the formula and simplified.

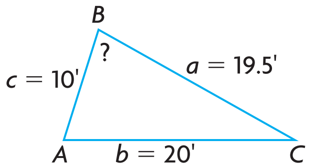
I had to isolate cos *B* before I could determine $\angle B$.

The angle formed at the peak of the roof is 78°. This plan will pass the local building code.

My answer is reasonable because $\angle B$ should be the angle with the largest measure in the triangle.

78° lies within the acceptable range of 70° to 80°.

Emilie's Solution: Rearranging the cosine law before substituting



I drew a diagram, labelling the sides and angles. I wrote all the side lengths in terms of feet.

$$b^2 = a^2 + c^2 - 2ac \cos B$$

Since I wanted to determine $\angle B$ and I knew the length of all three sides, I wrote the form of the cosine law that contains $\angle B$.

$$\begin{aligned}
b^2 + 2ac \cos B &= a^2 + c^2 - 2ac \cos B + 2ac \cos B \\
b^2 + 2ac \cos B &= a^2 + c^2 \\
b^2 + 2ac \cos B - b^2 &= a^2 + c^2 - b^2 \\
2ac \cos B &= a^2 + c^2 - b^2 \\
\frac{2ac \cos B}{2ac} &= \frac{a^2 + c^2 - b^2}{2ac} \\
\cos B &= \frac{a^2 + c^2 - b^2}{2ac}
\end{aligned}$$

I decided to rearrange the formula to solve for $\cos B$ by adding $2ac \cos B$ to both sides of the equation. Then I subtracted b^2 from both sides. Finally I divided both sides by $2ac$.

$$\cos B = \frac{19.5^2 + 10^2 - 20^2}{2(19.5)(10)}$$

I substituted the information that I knew into the rearranged formula and evaluated the right side.

$$\begin{aligned}
\cos B &= \frac{80.25}{390} \\
\cos B &= 0.2057...
\end{aligned}$$

$$\begin{aligned}
\angle B &= \cos^{-1}(0.2057...) \\
\angle B &= 78.125...°
\end{aligned}$$

The angle formed at the peak of the roof is 78°. This plan passes the local building code.

I rounded to the nearest degree. The value of this angle is within the acceptable range.

Your Turn

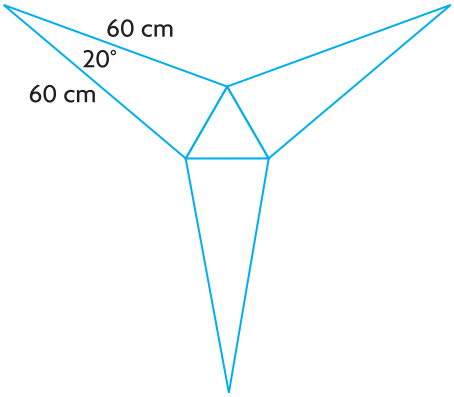
a) Compare Luanne's Solution and Emilie's Solution. What are the advantages of each strategy?

b) Which strategy do you prefer for this problem? Explain.

c) Use your strategy and the cosine law to determine $\angle A$ in $\triangle ABC$ above.

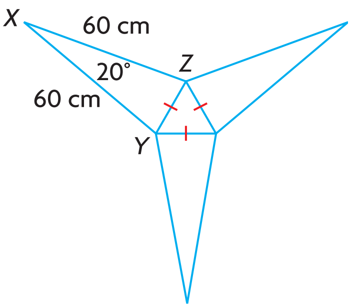
EXAMPLE 3 Solving a problem using the cosine law

A three-pointed star is made up of an equilateral triangle and three congruent isosceles triangles. Determine the length of each side of the equilateral triangle in this three-pointed star. Round the length to the nearest centimetre.



The world's largest Ukrainian Easter egg (called a pysanka) is located in Vegreville, Alberta. It is decorated with 2208 equilateral triangles and 524 three-pointed stars.

Dakoda's Solution



I named the vertices of one of the isosceles triangles.

$$(YZ)^2 = (XY)^2 + (XZ)^2 - 2(XY)(XZ)\cos(\angle YXZ)$$
$$(YZ)^2 = 60^2 + 60^2 - 2(60)(60)\cos 20°$$
$$(YZ)^2 = 3600 + 3600 - 6765.786...$$

I knew two sides and the contained angle in each isosceles triangle, so I used the cosine law to write an equation that involved YZ. Then I substituted the information that I knew.

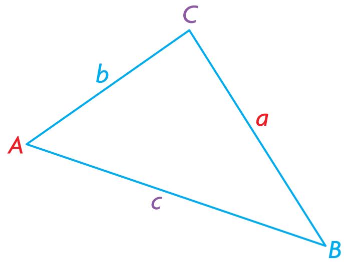
$$(YZ)^2 = 434.213...$$
$$YZ = \sqrt{434.213...}$$
$$YZ = 20.837...$$

Each side of the equilateral triangle has a length of 21 cm.

In Summary

Key Idea

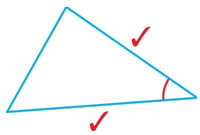
- The cosine law can be used to determine an unknown side length or angle measure in an acute triangle.

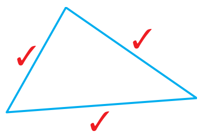


$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Need to Know

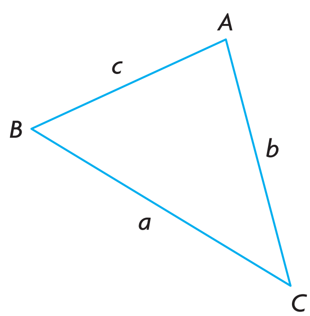
- You can use the cosine law to solve a problem that can be modelled by an acute triangle when you know:
  - two sides and the contained angle.
  - all three sides.





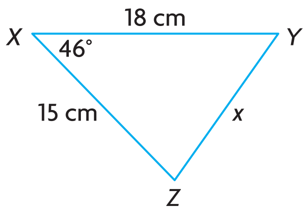
- The contained angle is the angle between two known sides.
- When using the cosine law to determine an angle, you can:
  - substitute the known values first, then solve for the unknown angle.
  - rearrange the formula to solve for the cosine of the unknown angle, then substitute and evaluate.

CHECK Your Understanding

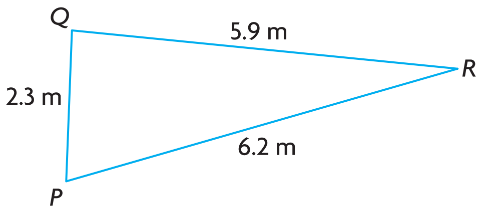


1. Suppose that you are given each set of data for $\triangle ABC$. Can you use the cosine law to determine c? Explain.
   a) $a = 5$ cm, $\angle A = 52°$, $\angle C = 43°$
   b) $a = 7$ cm, $b = 5$ cm, $\angle C = 43°$

2. Determine the length of side x to the nearest centimetre.



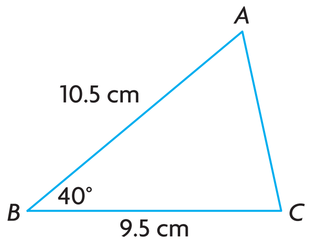
3. Determine the measure of $\angle P$ to the nearest degree.



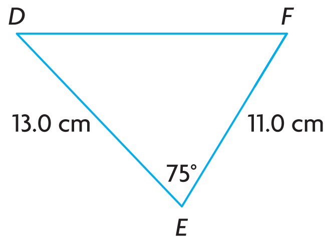
PRACTISING

4. Determine each unknown side length to the nearest tenth of a centimetre.

a)

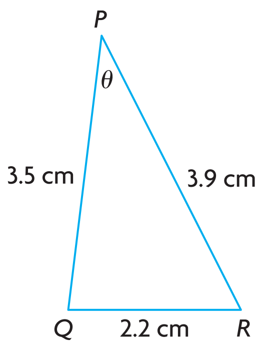


b)

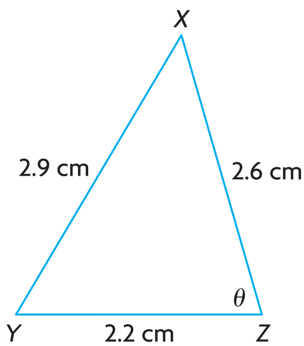


5. Determine the measure of each indicated angle to the nearest degree.

a)



b)



6. Sketch each triangle, based on the given equation. Then solve for the unknown side length or angle measure. Round all answers to the nearest tenth of a unit.

a) $w^2 = 15^2 + 16^2 - 2(15)(16) \cos 75°$

b) $k^2 = 32^2 + 35^2 - 2(32)(35) \cos 50°$

c) $48^2 = 46^2 + 45^2 - 2(46)(45) \cos Y$

d) $13^2 = 17^2 + 15^2 - 2(17)(15) \cos G$

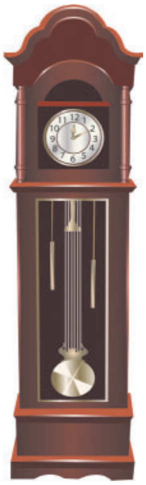
7. Solve each triangle. Round all answers to the nearest tenth of a unit.

a) In $\triangle DEF$, $d = 5.0$ cm, $e = 6.5$ cm, and $\angle F = 65°$.

b) In $\triangle PQR$, $p = 6.4$ m, $q = 9.0$ m, and $\angle R = 80°$.

c) In $\triangle LMN$, $l = 5.5$ cm, $m = 4.6$ cm, and $n = 3.3$ cm.

d) In $\triangle XYZ$, $x = 5.2$ mm, $y = 4.0$ mm, and $z = 4.5$ mm.



8. The pendulum of a grandfather clock is 100.0 cm long. When the pendulum swings from one side to the other side, the horizontal distance it travels is 9.6 cm.

a) Draw a diagram of the situation.

b) Determine the angle through which the pendulum swings. Round your answer to the nearest tenth of a degree.

9. Determine the perimeter of $\triangle SRT$, if $\angle S = 60°$, $r = 15$ cm, and $t = 20$ cm. Round your answer to the nearest tenth of a centimetre.

10. A parallelogram has sides that are 8 cm and 15 cm long. One of the angles in the parallelogram measures 70°. Explain how you could determine the length of the shorter diagonal.

11. a) A clock has a minute hand that is 20 cm long and an hour hand that is 12 cm long. Determine the distance between the tips of the hands at

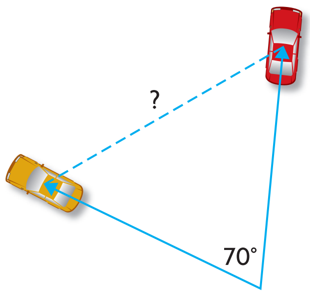
i) 2:00. **ii)** 10:00.

b) Discuss your results for part a).

12. Emilie makes stained glass windows to sell at the Festival du Bois in Maillardville, British Columbia. Each piece of glass is surrounded by lead edging. Emilie claims that she can create an acute triangle in part of a window using pieces of lead that are 15 cm, 36 cm, and 60 cm. Is she correct? Justify your decision.

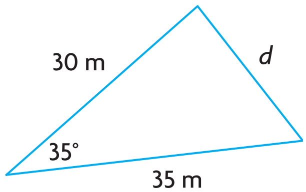
The Festival du Bois, one of British Columbia's greatest celebrations of French-Canadian culture, is held in March.

13. Two drivers leave their school at the same time and travel on straight roads that diverge by 70°. One driver travels at an average speed of 33.0 km/h. The other driver travels at an average speed of 45.0 km/h. How far apart will the two drivers be after 45 min, to the nearest tenth of a kilometre?



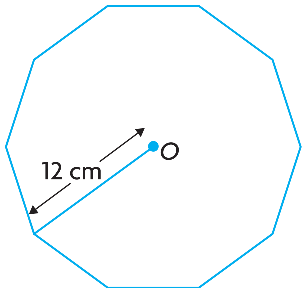
Closing

14. Use the triangle at the right to create a problem that involves side lengths and interior angles. Then describe how to determine the length of side *d*. Exchange your problem with a classmate.



Extending

15. The distance from the centre, *O*, of a regular decagon to each vertex is 12 cm. Determine the area of the decagon. Round your answer to the nearest square centimetre.



16. The centre, *O*, of a regular pentagon is a perpendicular distance of 1.5 cm from each side. Determine the perimeter, to the nearest tenth of a centimetre, and area, to the nearest tenth of a square centimetre, of the pentagon.

17. An ulu is an Inuit all-purpose knife, traditionally used by women. The metal blade of one type of ulu is roughly triangular in shape, with the cutting edge opposite the vertex where the handle is attached. The other sides of the ulu are roughly equal. Describe a functional ulu that has a 14 cm blade, measured point to point. Include the vertex angle at the handle and the side lengths in your description.