3.4 Solving Problems Using Acute Triangles

YOU WILL NEED
- ruler
- calculator

GOAL

Solve problems using the primary trigonometric ratios and the sine and cosine laws.

EXPLORE...
- Two planes leave an airport on different runways at the same time. One heads S40°W and the other heads S60°E. Create a problem about the planes that can be solved only by using the cosine law. Solve the problem.

LEARN ABOUT the Math

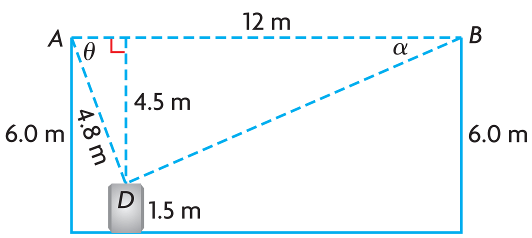
Two security cameras in a museum must be adjusted to monitor a new display of fossils. The cameras are mounted 6 m above the floor, directly across from each other on opposite walls. The walls are 12 m apart. The fossils are displayed in cases made of wood and glass. The top of the display is 1.5 m above the floor. The distance from the camera on the left to the centre of the top of the display is 4.8 m. Both cameras must aim at the centre of the top of the display.

? What is the angle of depression for each camera?

EXAMPLE 1 Connecting an acute triangle model to a situation

Determine the **angles of depression**, to the nearest degree, for each camera.

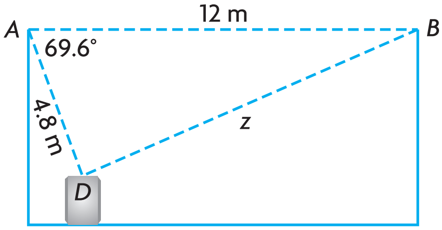
Vlad's Solution: Using primary trigonometric ratios and the cosine law



I drew a diagram. I placed the cameras 6 m from the floor, 12 m away from each other on opposite walls at points *A* and *B*. I wasn't sure where to place the display or its centre, *D*. The display had to be closer to camera A since the distance from camera A to the display was only 4.8 m. Subtracting the display height from 6 m gave me the distance from the display to the horizontal between the cameras. I labelled the angles of depression using θ and α.

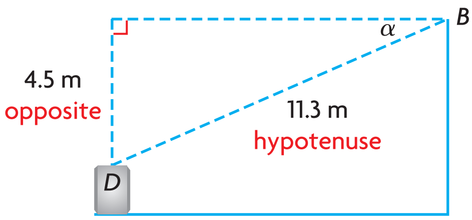
$$\sin\theta = \frac{4.5}{4.8}$$
$$\theta = \sin^{-1}\left(\frac{4.5}{4.8}\right)$$
$$\theta = 69.635...°$$

In the right triangle containing angle θ, the side of length 4.5 m is opposite angle θ, and the side of length 4.8 m is the hypotenuse. I could use the sine ratio to determine the measure of θ.



I knew the lengths of two sides in $\triangle ABD$ and the angle between them. So, I was able to use the cosine law to determine the length of the side opposite camera A, which I labelled z.

$$z^2 = 12^2 + 4.8^2 - 2(12)(4.8)\cos 69.635...°$$
$$z^2 = 126.952...$$
$$z = \sqrt{126.952...}$$
$$z = 11.267...$$

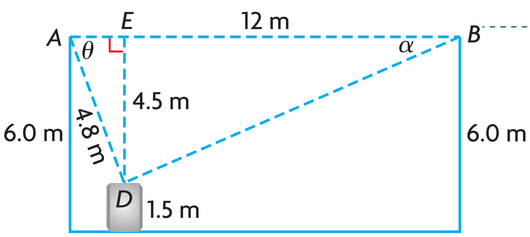


$$\sin\alpha = \frac{4.5}{11.267...}$$
$$\alpha = \sin^{-1}\left(\frac{4.5}{11.267...}\right)$$
$$\alpha = 23.539...°$$

Based on the sides I knew in the right triangle containing angle α, I wrote an equation using the sine ratio.

To monitor the display effectively, camera A must be adjusted to an angle of depression of 70° and camera B must be adjusted to an angle of depression of 24°.

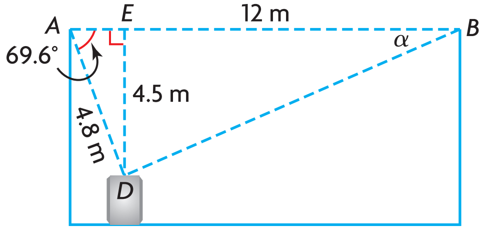
Michel's Solution: Using only primary trigonometric ratios



I drew a diagram by placing the cameras at points A and B and the centre of the display at point D. I knew point D had to be closer to camera A because the distance between it and camera B had to be greater than 4.8 m. Subtracting the display height from the camera height gave me the length of DE, the height of $\triangle ABD$. I labelled the angles of depression θ and α.

$$\sin \theta = \frac{4.5}{4.8}$$
$$\theta = \sin^{-1}\left(\frac{4.5}{4.8}\right)$$
$$\theta = 69.635...°$$

In right triangle *ADE*, *DE* is opposite angle θ and *AD* is the hypotenuse. Since I knew the lengths of both sides, I used the sine ratio to determine angle θ.

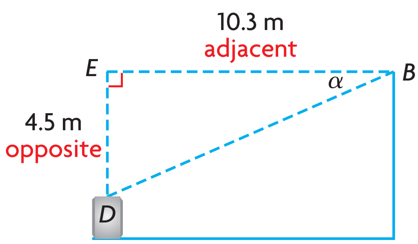


$$\cos 69.635...° = \frac{AE}{4.8}$$
$$4.8\,(\cos 69.635...°) = AE$$
$$1.670... = AE$$

In right triangle *ADE*, *AE* is adjacent to angle θ and *AD* is the hypotenuse. Since I knew the length of *AD* and the measure of angle θ, I used the cosine ratio to determine the length of *AE*.

$$EB = 12 - 1.670...$$
$$EB = 10.329...$$

To determine the length of *EB*, I subtracted the length of *AE* from 12.



Based on the sides I knew in right triangle *DEB*, I wrote an equation using the tangent ratio to determine angle α.

$$\tan \alpha = \frac{4.5}{10.329...}$$
$$\alpha = \tan^{-1}\left(\frac{4.5}{10.329...}\right)$$
$$\alpha = 23.539...°$$

Camera A must be adjusted to an angle of depression of 70° and camera B must be adjusted to an angle of depression of 24° to ensure that they both point to the centre of the display.

Reflecting

A. Why do you think Vlad started his solution by using the right triangle that contained angle θ instead of the right triangle that contained angle α?

B. Could Vlad have determined the value of α using the sine law? Explain.

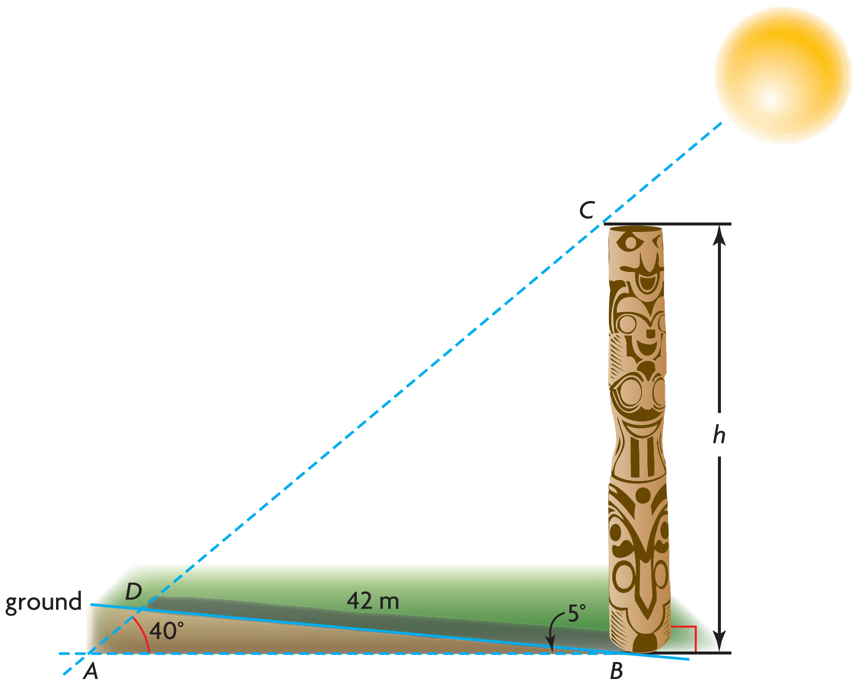
C. Which solution do you prefer? Justify your choice.

APPLY the Math

EXAMPLE 2 Connecting acute triangle models to indirect measurement

The world's tallest free-standing totem pole is located in Beacon Hill Park in Victoria, British Columbia. It was carved from a single cedar log by noted carver Chief Mungo Martin of the Kwakiutl (Kwakwaka'wakw), with a team that included his son David and Henry Hunt. It was erected in 1956.

While visiting the park, Manuel wanted to determine the height of the totem pole, so he drew a sketch and made some measurements:



- I walked along the shadow of the totem pole and counted 42 paces, estimating each pace was about 1 m.
- I estimated that the **angle of elevation** of the Sun was about 40°.
- I observed that the shadow ran uphill, and I estimated that the angle the hill made with the horizontal was about 5°.

How can Manuel determine the height of the totem pole to the nearest metre?

Manuel's Solution

h is the height of $\triangle ABC$, but it is also a side in acute triangle DCB.

I needed to determine the angles in this triangle to be able to determine h using the sine law.

$$\angle ADB = 180° - 40° - 5°$$
$$\angle ADB = 135°$$

I used the two angles I knew in $\triangle ADB$ to determine the third angle in this triangle, since the angles in a triangle add to 180°.

$$\angle CDB = 180° - 135°$$
$$\angle CDB = 45°$$

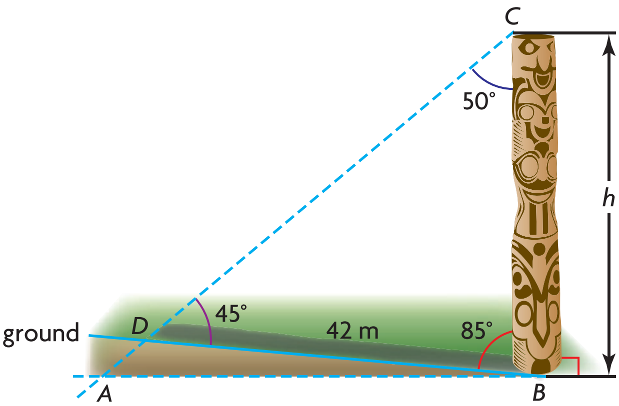
I subtracted $\angle ADB$ from 180° to determine $\angle CDB$, since these are supplementary angles.

$$\angle CBD = 90° - 5°$$
$$\angle CBD = 85°$$

I subtracted $\angle DBA$ from 90° to determine $\angle CBD$, since these are complementary angles.

$$\angle DCB = 180° - 45° - 85°$$
$$\angle DCB = 50°$$

I used the two angles I determined in $\triangle DCB$ to determine the third angle in this triangle.



I added all the information I determined about $\triangle DCB$ to my sketch.

$$\frac{h}{\sin 45°} = \frac{42}{\sin 50°}$$

$$\sin 45°\left(\frac{h}{\sin 45°}\right) = \sin 45°\left(\frac{42}{\sin 50°}\right)$$

$$h = \sin 45°\left(\frac{42}{\sin 50°}\right)$$

$$h = 38.768...$$

I used the sine law to write an equation that contained h. Then I solved for h.

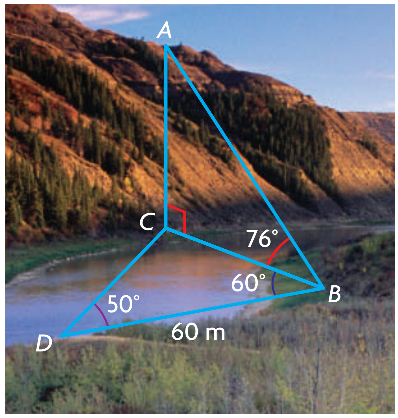
The totem pole is 39 m tall.

Your Turn

List some sources of error that may have occurred in Manuel's strategy that would affect the accuracy of his determination.

EXAMPLE 3 Solving a three-dimensional problem

Brendan and Diana plan to climb the cliff at Dry Island Buffalo Jump, Alberta. They need to know the height of the climb before they start. Brendan stands at point B, as shown in the diagram. He uses a clinometer to determine $\angle ABC$, the angle of elevation to the top of the cliff. Then he estimates $\angle CBD$, the angle between the base of the cliff, himself, and Diana, who is standing at point D. Diana estimates $\angle CDB$, the angle between the base of the cliff, herself, and Brendan.



Determine the height of the cliff to the nearest metre.

Diana's Solution

In $\triangle DBC$,

$$\angle BCD = 180° - 60° - 50°$$
$$\angle BCD = 70°$$

I didn't have enough information about $\triangle ABC$ to determine the height, AC. I needed the length of BC. BC is in $\triangle ABC$, but it is also in $\triangle DBC$.

I knew two angles and a side length in $\triangle DBC$. Before I could determine BC, I had to determine $\angle BCD$. I used the fact that the sum of all three interior angles is 180°.

$$\frac{BC}{\sin D} = \frac{BD}{\sin C}$$
$$\frac{BC}{\sin 50°} = \frac{60}{\sin 70°}$$

I used the sine law to write an equation that involved BC in $\triangle DBC$.

$$\sin 50°\left(\frac{BC}{\sin 50°}\right) = \sin 50°\left(\frac{60}{\sin 70°}\right)$$

To solve for BC, I multiplied both sides of the equation by sin 50°.

$$BC = \sin 50°\left(\frac{60}{\sin 70°}\right)$$
$$BC = 48.912...$$

$$\tan 76° = \frac{AC}{BC}$$
$$\tan 76° = \frac{AC}{48.912...}$$

I knew that $\triangle ABC$ is a right triangle. I also knew that AC is opposite the 76° angle and BC is adjacent to it. So, I used the tangent ratio to write an equation that involved AC.

$$48.912...(\tan 76°) = 48.912...\left(\frac{AC}{48.912...}\right)$$
$$196.177... = AC$$

The height of the cliff is 196 m.

Your Turn

Create a three-dimensional problem that can be solved using Diana's strategy. What features of your problem make it necessary to use two triangles to solve the problem?

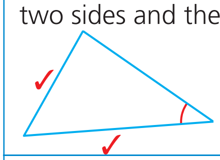
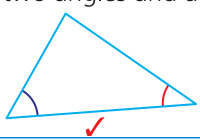
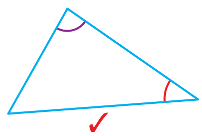
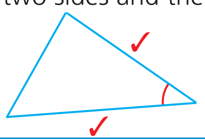
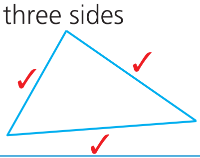
In Summary

Key Idea

- The sine law, the cosine law, the primary trigonometric ratios, and the sum of angles in a triangle may all be useful when solving problems that can be modelled using acute triangles.

Need to Know

- To decide whether you need to use the sine law or the cosine law, consider the information given about the triangle and the measurement to be determined.

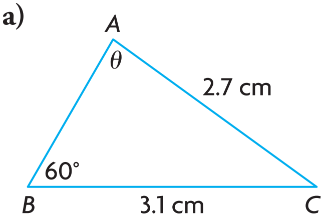
Information Given	Measurement to be Determined	Use
two sides and the angle opposite one of the sides 	angle	sine law
two angles and a side  	side	sine law
two sides and the contained angle 	side	cosine law
three sides 	angle	cosine law

- Drawing a clearly labelled diagram makes it easier to select a strategy for solving a problem.

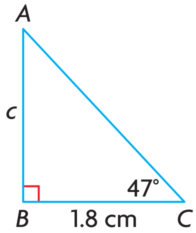
CHECK Your Understanding

1. Explain how you would determine the indicated angle measure or side length in each triangle.

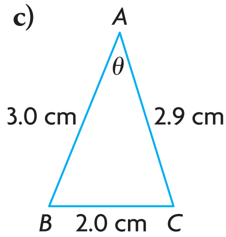
a)



b)



c)



2. a) Use the strategies you described to determine the measurements indicated in question 1. Round your answers to the nearest tenth of a unit.

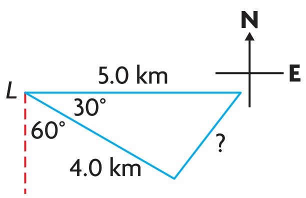
b) Compare your answers for questions 1 and 2a) with a classmate's answers. Which strategy seems to be most efficient for each?

PRACTISING

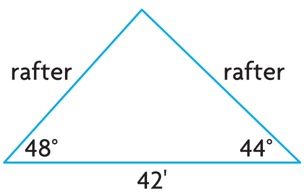
3. A kayak leaves Rankin Inlet, Nunavut, and heads due east for 5.0 km, as shown in the diagram. At the same time, a second kayak travels in a direction S60°E from the inlet for 4.0 km. How far apart, to the nearest tenth of a kilometre, are the kayaks?

a) Describe how you can solve the problem.

b) Determine the distance between the kayaks.



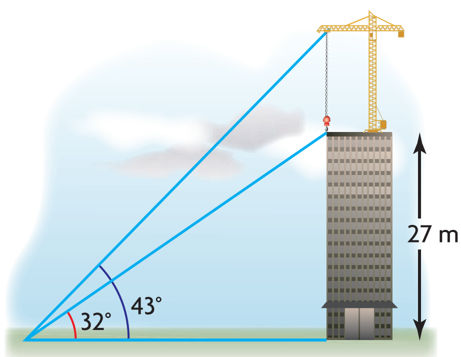
4. How long, to the nearest inch, is each rafter in the roof shown?



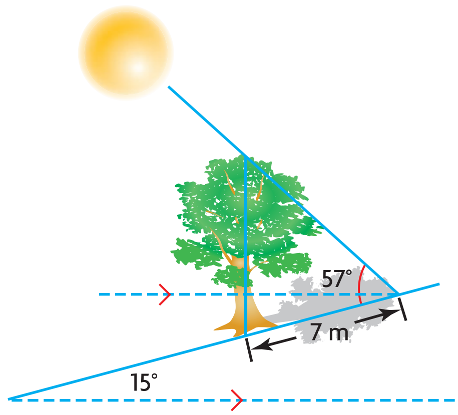
5. A crane stands on top of a building, as shown.

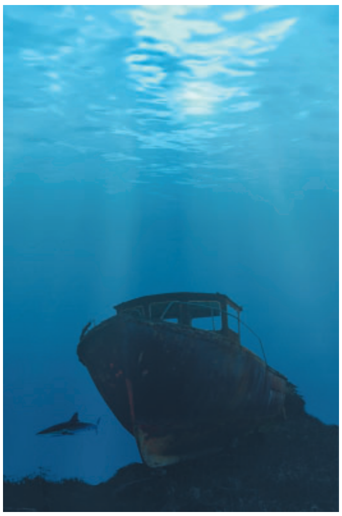
a) How far is the point on the ground from the base of the building, to the nearest tenth of a metre?

b) How tall is the crane?



6. A tree is growing on a hillside, as shown. The hillside is inclined at an angle of 15° to the horizontal. The tree casts a shadow uphill. How tall is the tree, to the nearest metre?
   a) Describe how you can solve the problem.
   b) Determine the height of the tree.





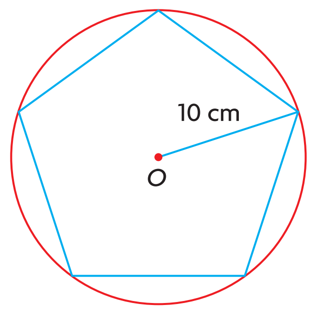
Ships are sometimes deliberately sunk (scuttled) to form breakwaters and artificial reefs.

7. A radar operator on a ship discovers a large sunken vessel lying parallel to the ocean surface, 200 m directly below the ship. The length of the vessel is a clue to which wreck has been found. The radar operator measures the angles of depression to the front and back of the sunken vessel to be 56° and 62°. How long, to the nearest tenth of a metre, is the sunken vessel?

8. Fred and Agnes are 520 m apart. As Brendan flies overhead in an airplane, they estimate the angle of elevation of the airplane. Fred, looking south, estimates the angle of elevation to be 60°. Agnes, looking north, estimates it to be 40°. What is the altitude of the airplane, to the nearest tenth of a metre?

9. Two support wires are fastened to the top of a communications tower from points *A* and *B* on the ground. The points are on opposite sides of the tower and in line. One wire is 18 m long, and the other wire is 12 m long. The angle of elevation of the longer wire to the top of the tower is 38°.
   a) How tall is the tower, to the nearest tenth of a metre?
   b) How far apart are points *A* and *B*, to the nearest tenth of a metre?

10. A regular pentagon is inscribed in a circle with centre *O*, as shown in the diagram.
   a) Work with a partner to develop a strategy to determine the perimeter of the pentagon.
   b) Carry out your strategy to determine the perimeter to the nearest tenth of a centimetre.



11. Ryan is in a police helicopter, 400 m directly above the Sea to Sky highway near Whistler, British Columbia. When he looks north, the angle of depression to a car accident is 65°. When he looks south, the angle of depression to the approaching ambulance is 30°.

a) How far away is the ambulance from the scene of the accident, to the nearest tenth of a metre?

b) The ambulance is travelling at 80 km/h. How long will it take the ambulance to reach the scene of the accident?

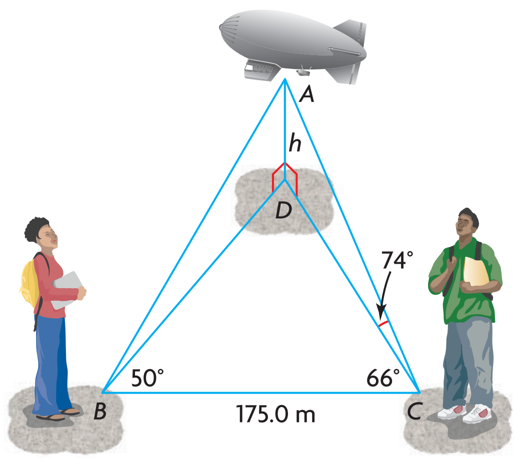
12. The radar screen in the air-traffic control tower at the Edmonton International Airport shows that two airplanes are at the same altitude. According to the range finder, one airplane is 100 km away, in the direction N60°E. The other airplane is 160 km away, in the direction S50°E.

a) How far apart are the airplanes, to the nearest tenth of a kilometre?

b) If the airplanes are approaching the airport at the same speed, which airplane will arrive first?

13. In a parallelogram, two adjacent sides measure 10 cm and 12 cm. The shorter diagonal is 15 cm. Determine, to the nearest degree, the measures of all four angles in the parallelogram.

14. Two students decided to determine the altitude, h, of a promotional blimp flying over McMahon Stadium in Calgary. The students' measurements are shown in the diagram. Determine h to the nearest tenth of a metre. Explain each of your steps.



Math in Action

How Good Is Your Peripheral Vision?

When you stare straight ahead, you can still see objects to either side. This is called peripheral vision. It can be measured using an angle. For example, the angle for your right eye would be swept out from a point directly in front of your nose to the point where you can no longer see objects on the far right. This angle is about 60° for those with normal peripheral vision.

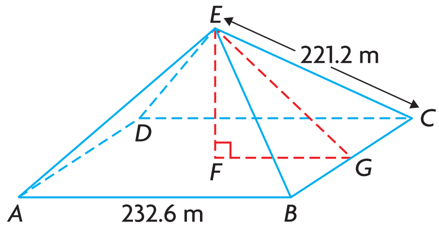
- Work with a partner or in a small group.
- Make a plan to measure the peripheral vision of your eyes. The only materials you can use are a pencil, a metre stick, and string.
- Test your plan. What is your peripheral vision?
- Evaluate your plan. What adjustments did you need to make during the test? Are you satisfied that your plan worked well? Explain.

Closing

15. Create a real-life problem that can be modelled by an acute triangle. Exchange the problem you created with a classmate. Sketch the situation in your classmate's problem, and explain what must be done to solve it.

Extending

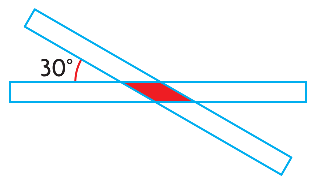
16. The Great Pyramid at Giza in Egypt has a square base with sides that are 232.6 m long. The distance from the top of the pyramid to each corner of the base was originally 221.2 m.



a) Determine the apex angle of a face of the pyramid (for example, $\angle AEB$) to the nearest degree.

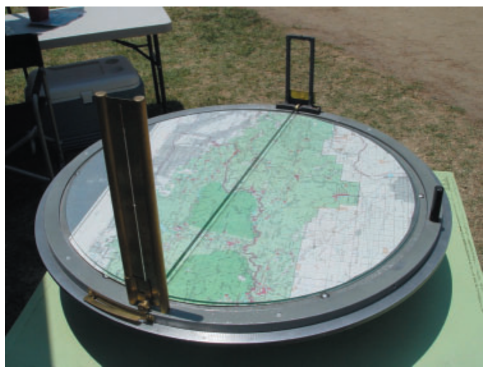
b) Determine the angle that each face makes with the base (for example, $\angle EGF$) to the nearest degree.

17. Cut out two paper strips, each 5 cm wide. Lay them across each other as shown at the right. Determine the area of the overlapping region. Round your answer to the nearest tenth of a square centimetre.



History | Connection

Fire Towers and Lookouts



For about 100 years, observers have been watching for forest fires. Perched in lookouts on high ground or on tall towers (there are three fire towers in Alberta that are 120 m high), the observers watch for smoke in the surrounding forest. When the observers see signs of a fire, they report the sighting to the Fire Centre.

The first observers may have had as little as a map, binoculars, and a horse to ride to the nearest station to make a report. Today, observers have instruments called alidades and report using radios. Alidades consist of a local map, fixed in place, with the tower or lookout at the centre of the map. Compass directions are marked in degrees around the edge of the map. A range finder rotates on an arc above the map. The observer rotates the range finder to get the smoke in both sights, notes the direction, and reports it to the Fire Centre. A computer at the Fire Centre can use the locations of the fire towers and directions from two or more observers to fix the location of the fire. The latitude and longitude are then entered into a helicopter's flight GPS system. The helicopter's crew flies to the fire, records a more accurate location, and reports on the fire.

A. How could a Fire Centre use trigonometry to determine the location of a fire?

B. How could an observer use trigonometry to estimate the size of a fire?

Applying Problem-Solving Strategies

Analyzing a Trigonometry Puzzle

Puzzles do not always have precise solutions. They cannot always be solved purely by deduction, although logic helps.

YOU WILL NEED

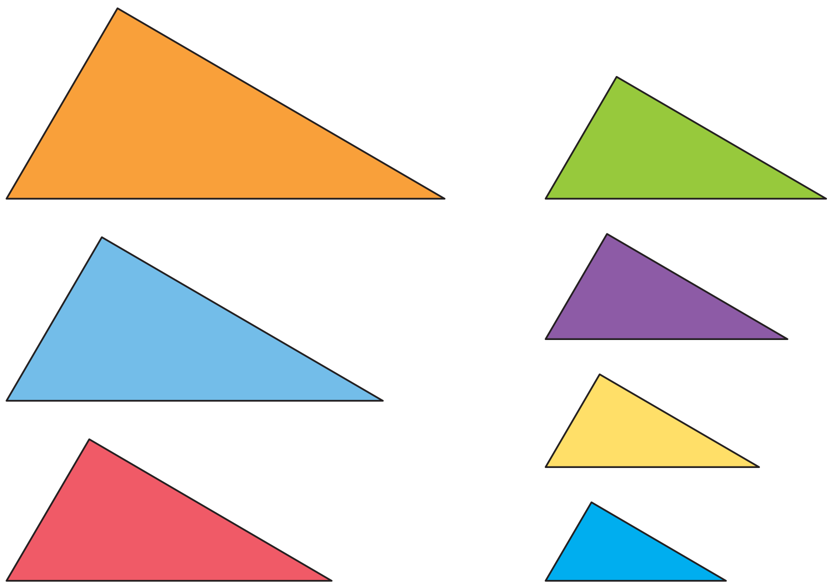
- ruler
- scissors

The Puzzle

A. Below are seven similar right triangles. Trace the triangles, and cut them out.

B. Use all seven triangles to form a single square, with no overlapping.

C. If the hypotenuse of the greatest triangle is 10 units long, what is the area of the square?



The Strategy

D. Describe the strategy you used to form the square.

E. Describe the strategy you used to determine the area of the square.