1.4 Applications of Set Theory

GOAL

Use sets to model and solve problems.

INVESTIGATE the Math

Rachel surveyed Grade 12 students about how they communicated with friends over the previous week.

- 66% called on a cellphone.
- 76% texted.
- 34% used a social networking site.
- 56% called on a cellphone and texted.
- 18% called on a cellphone and used a social networking site.
- 19% texted and used a social networking site.
- 12% used all three forms of communication.

EXPLORE…

Sarah conducted a survey of teen gaming preferences. Here are her results:

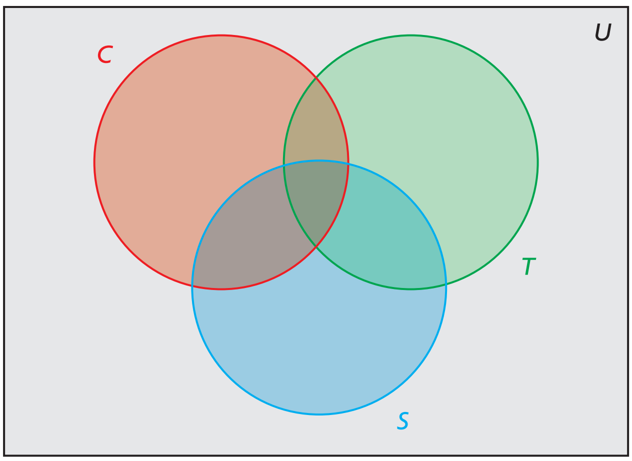
- 20 teens play online games.
- 20 play on a game console.
- 20 play games on their cellphone.

She surveyed only 31 teens. How can this be?

? What percent of students used at least one of these three forms of communication?

A. The Venn diagram below represents the following sets:

- C = {students who called on a cellphone}
- T = {students who texted}
- S = {students who used a social networking site}



i) What does the universal set U represent in this situation?

ii) Copy this Venn diagram. Record the percent of students who used all three forms of communication on your diagram.

B. Determine the percent of students who texted and used a social networking site, but did not call on a cellphone. Update your diagram.

C. Determine the percent of students who called on a cellphone and used a social networking site, but did not text. Determine the percent of students who called on a cellphone and texted, but did not use a social networking site. Update your diagram.

D. Determine the percent of students who only called on a cellphone, only texted, or only used a social networking site. Update your diagram.

E. Determine the percent of students who used at least one of these three forms of communication. Explain your answer.

Reflecting

F. Serge claims that the Principle of Inclusion and Exclusion can be used to develop a formula for $n(S \cup T \cup C)$ as follows:

$$n(S \cup T \cup C) = n(S) + n(T) + n(C) - n(S \cap T) - n(S \cap C) - n(T \cap C) + n(S \cap T \cap C)$$

Does this formula give the same answer you got in part E? Explain.

G. Determine the percent of students who called on a cellphone or texted, but did not use a social networking site. Express your result in set notation.

H. How would your Venn diagram change if 16% of the students had used all three forms of communication?

APPLY the Math

EXAMPLE 1 Solving a puzzle using the Principle of Exclusion and Inclusion

Use the following clues to answer the questions below:

- 28 children have a dog, a cat, or a bird.
- 13 children have a dog.
- 13 children have a cat.
- 13 children have a bird.
- 4 children have only a dog and a cat.
- 3 children have only a dog and a bird.
- 2 children have only a cat and a bird.
- No child has two of each type of pet.

a) How many children have a cat, a dog, and a bird?
b) How many children have only one pet?

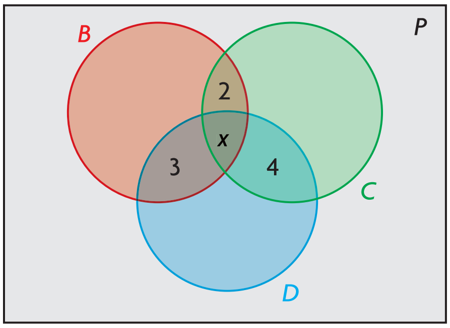
Hailey's Solution: Using the Principle of Inclusion and Exclusion

a) $P = \{\text{children with pets}\}$ $\quad C = \{\text{children with a cat}\}$
$B = \{\text{children with a bird}\}$ $\quad D = \{\text{children with a dog}\}$

I defined the sets in this situation.

Let x represent the number of children with a bird, a cat, and a dog.

I defined the variable x.



I drew a Venn diagram to show how the numbers of elements in the four sets were related.

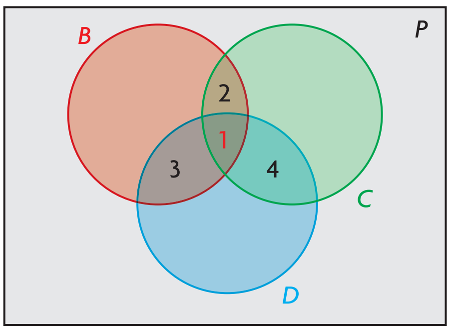
$$n(B) + n(C) + n(D) - n(B \cap C) - n(B \cap D) - n(C \cap D) + n(B \cap C \cap D) = n(B \cup C \cup D)$$

Each circle overlaps the other two circles. I determined the sum of the three circles using the Principle of Inclusion and Exclusion to deal with the overlapping areas.

$$13 + 13 + 13 - (x + 2) - (x + 3) - (x + 4) + x = 28$$
$$39 - x - 2 - x - 3 - x - 4 + x = 28$$
$$30 - 2x = 28$$
$$-2x = -2$$
$$x = 1$$

I knew that each circle represents 13 children, and there are 28 children in total. I solved for x.

One child has three different types of pets.



The area of the Venn diagram where all three circles overlap represents children who have all three pets. I revised my diagram.

b) Children with one pet = Total number of children − Children who have more than one pet

Children with one pet $= 28 - (1 + 2 + 3 + 4)$
Children with one pet $= 28 - 10$
Children with one pet $= 18$

I subtracted the number of children who have more than one pet from the total number of children.

Therefore, 18 children have only one type of pet.

Your Turn

Suppose that the first clue is changed so there are 24 children who have a dog, a cat, or a bird. All the other clues remain the same. Determine the number of children who have all three types of pet and the number of children who have only one type of pet.

EXAMPLE 2 Searching on the Internet

Hillary and Liam are working on a project for their World Issues class. They need to use the Internet to gather information about popular culture, especially social criticism on television. Liam suggests they search for popular culture using a search engine.

How can they refine their search to narrow down the number of hits?

Hillary's Solution

U = {all sites on the Internet}

R = {sites containing the words *popular* and *culture*}

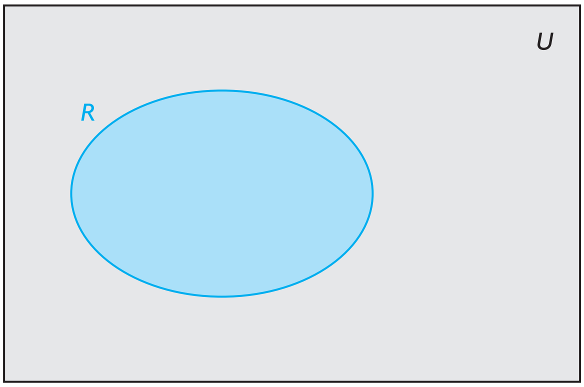
I defined the universal set and the set that Liam suggested searching.

popular culture

About 186,000,000 results (0.08 seconds)

The results included all the sites with the words *popular* and *culture*, in any order and not necessarily together.

These are too many results for me to analyze.



I drew a Venn diagram to illustrate my search.

"popular culture"

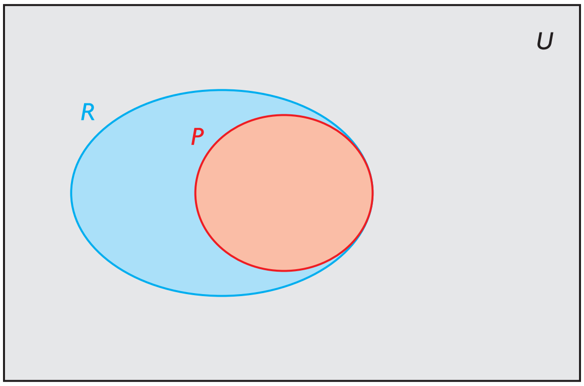
About 6,520,000 results (0.08 seconds)

> I refined my search using quotation marks, so the results included only sites with the exact phrase *"popular culture."*

I got fewer results, but still too many to analyze.

P = {sites containing the phrase "*popular culture*"}

> I defined set P.



> I revised my Venn diagram.
>
> I did not draw the shapes to scale. The shapes just show the relationship between the two sets. Set P is a subset of set R.

The second search is a subset of the first search.

"popular culture" and television

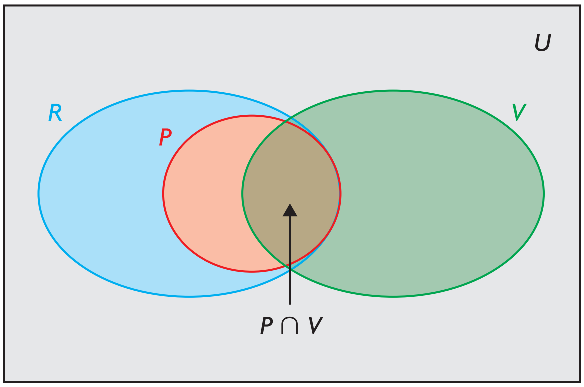
About 2,570,000 results (0.30 seconds)

> I searched *"popular culture"* and *television* together. I used "and" so I could locate articles that contained both.

I got even fewer results, but still too many.

V = {all *television* sites}

> I defined set V and revised my diagram.



> The area of my diagram that contains both *"popular culture"* and *television* is the intersection of sets P and V.

"popular culture" and "television shows"

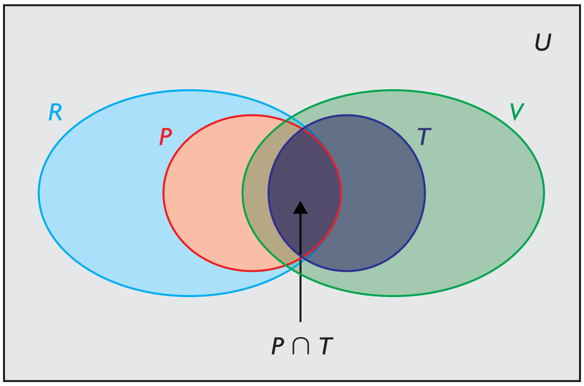
About 105,000 results (0.15 seconds)

> Some sites were not about television shows, so I refined my search. I put quotation marks around *"television shows"* to search for the exact phrase.

It's getting better.

$T = \{$"*television shows*" sites$\}$

I defined set *T* and revised my diagram. Set *T* is a subset of set *V*.



The area of my diagram that contains both "*popular culture*" and "*television shows*" is the intersection of sets *P* and *T*.

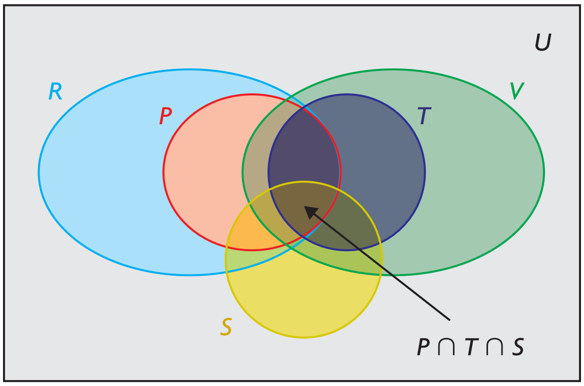
"popular culture" and "television shows" and "social criticism"

About 962 results (0.20 seconds)

I included "*social criticism*" in quotation marks to narrow down the number of hits.

$S = \{$"*social criticism*" sites$\}$

I defined set *S* and revised my diagram. Set *S* intersects the other sets.



The area of my diagram that contains "*popular culture*," "*television shows*," and "*social criticism*" is the intersection of sets *P*, *T*, and *S*.

I had about 962 results. I can deal with this number of results.

I narrowed my search from 186 million hits to only 962 hits.

$P \cap T \cap S = \{$popular culture and social criticism on television shows$\}$

$n(P \cap T \cap S) = 962$

Since my final search included all three criteria, it consisted of the intersection of all three sets.

Your Turn

Liam knows that animated television shows often use humour to comment on serious social issues. How would including the word *animated* in an Internet search affect the number of hits? Explain.

EXAMPLE 3 Correcting errors that involve sets

Shannon's high school starts a campaign to encourage students to use "green" transportation for travelling to and from school. At the end of the first semester, Shannon's class surveys the 750 students in the school to see if the campaign is working. They obtain these results:

- 370 students use public transit.
- 100 students cycle and use public transit.
- 80 students walk and use public transit.
- 35 students walk and cycle.
- 20 students walk, cycle, and use public transit.
- 445 students cycle or use public transit.
- 265 students walk or cycle.

How many students use green transportation for travelling to and from school?

Alaina solved this problem as shown below, but she made some errors. What errors did she make? Determine the correct solution.

Alaina's Solution

445 students cycle or use public transit.
370 use public transit.

Therefore, 445 − 370 or 75 students cycle but do not walk or use public transit.

> I knew how many students use public transit or cycle and how many use public transit, so I subtracted to determine the number who only cycle.

265 walk or cycle.
75 students just cycle.

Therefore, 265 − 75 or 190 students only walk.

> I knew how many students cycle and how many walk or cycle. I used this information to determine the number of students who only walk.

370 students use public transit.
75 students only cycle.
190 students only walk.
100 students cycle and use public transit.
80 students walk and use public transit.
35 students walk and cycle.
20 students use all three methods of green transportation.

> I listed the number of students in each category.

Therefore, 370 + 75 + 190 + 100 + 80 + 35 + 20 or 870 students use green transportation.

> I added the numbers in each category. I knew that my answer is wrong because there are only 750 students in the school.

Alberto's Solution

When determining how many students cycle, Alaina did not account for the number of students who both cycle and use public transit, or for the number of students who use all three methods of transportation. She made the same mistake when determining how many students walk.

I examined Alaina's solution and identified the errors she made.

Let U represent the universal set:

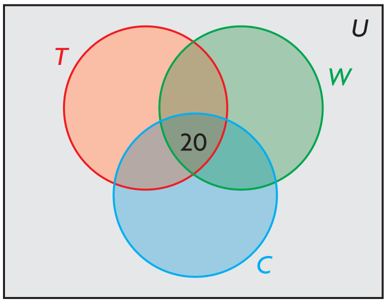
U = {students who attend Shannon's school}

T = {students who use public transit}

W = {students who walk}

C = {students who cycle}

I defined the sets in this situation.



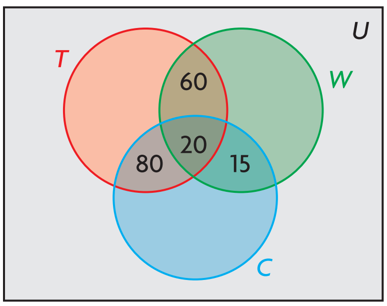
I drew a Venn diagram that showed the number of elements in each region.

I knew that 20 students use all three methods of transportation. I entered 20 where the three circles intersect.

The number who walk and use public transit is 80. Of those students, 20 have already been counted (the number who do both of those things and also cycle). That leaves $80 - 20$ or 60 to go in the other region in the intersection of T and W.

The number who walk and cycle is 35. Of those students, 20 have already been counted. That leaves $35 - 20$ or 15 to go in the other region of the intersection of C and W.

The number who cycle and use public transit is 100. Of those students, 20 have already been counted. That leaves $100 - 20$ or 80 to go in the other part of the intersection of T and C.



The number of students who use public transit is 370. Of those, 60 + 20 + 80 or 160 have already been counted. The number who use only public transit is 370 − 160 or 210.

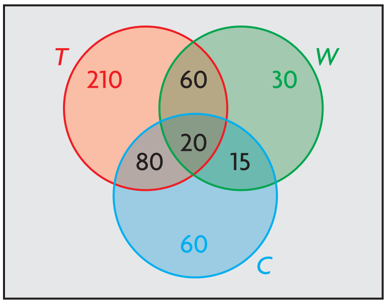
I used the same reasoning to determine the number of students who only use public transit, or walk, or only cycle.

The number of students who cycle or use public transit is 445. Of those, 370 use public transit. The number who cycle but don't use public transit is 445 − 370 or 75.

I knew that "or" means the union of two sets.

Of the 75 students who cycle but don't use public transit, 15 have already been counted. The number who only cycle is 75 − 15 or 60.

The number of students who walk or cycle is 265. Of those, 60 + 80 + 15 + 20 + 60 or 235 have already been counted. The number who only walk is 265 − 235 or 30.



$n(T) = 210 + 60 + 20 + 80$
$n(T) = 370$ ✓

$n(C \cup T) = 60 + 15 + 370$
$n(C \cup T) = 445$ ✓

$n(W \cup C) = 60 + 20 + 15 + 30 + 80 + 60$
$n(W \cup C) = 265$ ✓

I checked my Venn diagram by verifying the number of students who use public transit, who cycle or use public transit, and who walk or cycle.

The numbers in my Venn diagram are reasonable.

Your Turn

How many students use exactly one method of green transportation?

EXAMPLE 4 Winning a game

Star is playing a game that involves sets. She is using the nine cards shown, which have three different attributes: shape, colour, and number of shapes. There are three shapes (triangle, square, and circle), three colours (red, blue, and green), and three numbers of shapes (one, two, and three).

To win, Star must create four sets, using three cards in each set from the nine cards shown. Each card may be used more than once in a set.

What sets can Star make to win the game?

Rules for Creating Sets

Sets of three cards must agree with each other or disagree with each other with respect to each attribute. Three cards form a set if

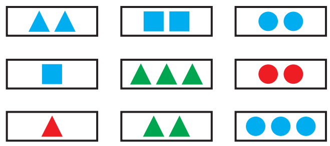
- the cards display the same number of figures or each displays a different number of figures, **and**
- the figures are all the same shape or three different shapes, **and**
- the figures are the same colour or three different colours.

For example:

Set 1:

Set 2:

Not a set:



Star's Solution

B = {blue cards} $n(B) = 5$

G = {green cards} $n(G) = 2$

R = {red cards} $n(R) = 2$

First, I examined the attribute colour.

I do not have enough green cards or red cards to make a set of three with the same colour.

I can make a set of blue cards with different numbers and shapes:
1 blue square, 2 blue triangles, and 3 blue circles.

I examined the blue cards.
One card has one shape.
Three cards have two shapes.
One card has three shapes.
Two cards have squares, two have circles, and one has triangles.

S = {square cards} $n(S) = 2$

I cannot make a set where all the shapes on the cards are squares.

I examined the attribute shape. Only two cards have squares, so I cannot make a set with all squares.

T = {triangle cards} $n(T) = 4$

T = {1 red, 2 blue, 2 green, 3 green}

Four cards have triangles. There are three different colours and numbers of triangles on these cards. By the rules, all the colours and all the numbers need to be different.

I can make a set with the same triangle shape:
1 red triangle, 2 blue triangles, and 3 green triangles.

$C = \{\text{circle cards}\}$ $\quad n(C) = 3$
$C = \{\text{2 blue, 2 red, 3 blue}\}$

There are only two different numbers of circles on the circle cards, and only two different colours. To make a set, the colours must be all the same or all different.

I cannot make a set where all the shapes on the cards are circles.

$O = \{\text{1-shape cards}\}$ $\quad n(O) = 2$
$W = \{\text{2-shape cards}\}$ $\quad n(W) = 5$
$H = \{\text{3-shape cards}\}$ $\quad n(H) = 2$

I examined the attribute number. I knew that I could not make a set with the 1- or 3-shape cards since I have only two of them.

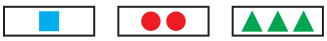
I cannot make a set where the number of shapes on each card is one or three.

I can make a set with the same number, 2:
2 blue squares, 2 green triangles, and 2 red circles.

To make a set with the same number, I used three different shapes and three different colours.

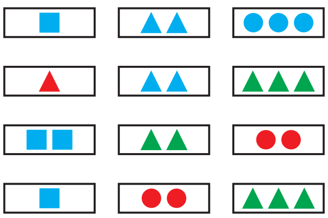
I can make a set where all the attributes are different.
1 blue square, 2 red circles, and 3 green triangles.

Finally, I looked for a set in which all three attributes were different.



I made four sets of three cards.

Set 1:
Set 2:
Set 3:
Set 4:

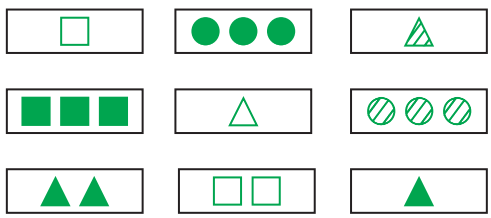


I won the game. My sets are:

Set 1: same colour, different shape, different number
Set 2: same shape, different colour, different number
Set 3: same number, different shape, different colour
Set 4: different shape, different colour, different number

Your Turn

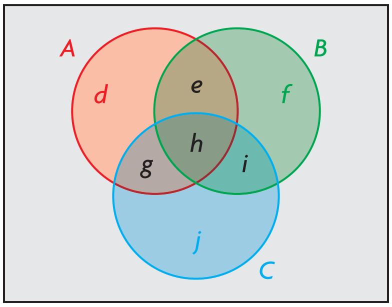
The following cards can be used for another game. Win this game by creating four sets of three cards, using the attributes of shape, number, and shading.



In Summary

Key Ideas

- Set theory is useful for solving many types of problems, including Internet searches, database queries, data analyses, games, and puzzles.
- To represent three intersecting sets with a Venn diagram, use three intersecting circles. For example, in the following Venn diagram,



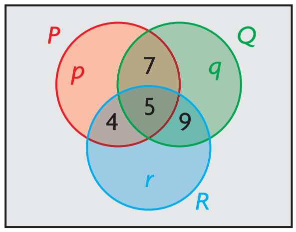
  - $A \cap B \cap C$ is represented by region h,
  - $A \cap B$ is represented by the union of regions e and h,
  - $A \cap C$ is represented by the union of regions g and h, and
  - $B \cap C$ is represented by the union of regions h and i.

  Each region of a Venn diagram contains elements that occur only in that particular region.
- You can use the Principle of Inclusion and Exclusion to determine the number of elements in the union of three sets:
  $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

Need to Know

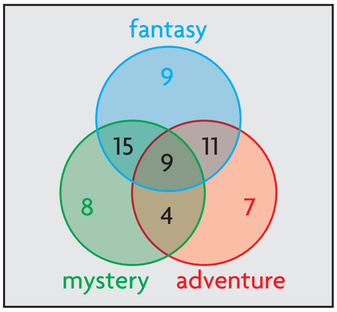
- You can use concepts related to sets to search for websites on the Internet:
  - Put an exact phrase in quotation marks.
  - Connect words or phrases with "and" to search for sites that contain both. The word "and" represents the intersection of two or more sets.
  - Connect words or phrases with "or" to search for sites that contain either one or the other, or both. The word "or" represents the union of two or more sets.
- When solving a puzzle or problem, it is often useful to visualize the problem. First identify which sets are defined by the context. Then identify how the sets overlap. Finally, identify regions of the overlaps that are of interest in the puzzle or problem. It is often advisable to consider how much is known about each region, and use the information about the region that is most known to deduce information about regions that are less well known. A systematic approach will result in answers that are easier to verify.

CHECK Your Understanding



1. The three circles in the Venn diagram (P, Q, and R) contain the same number of elements. Determine one set of values for p, q, and r.

2. The members of a book club read fantasy, mystery, and adventure books. The following Venn diagram shows the types of books that the members like:



Use the diagram to determine each amount below.

a) $n((F \cup M) \setminus A)$
b) $n((A \cup F) \setminus M)$
c) $n((F \cup A) \cup (F \cup M))$
d) $n(A \setminus F \setminus M)$

PRACTISING

3. Someone left a backpack full of school books on a transit bus. The only identification is the name "David Smith," so the bus driver takes the backpack to the public school board office. The staff search their database and learn that 56 students have this name. How can the staff narrow their search using search tools and other items in the backpack?

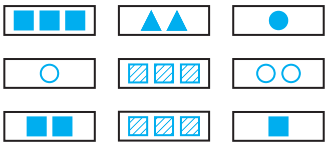
4. Jennifer is an optician. She is trying to decide whether she should offer a package deal to customers who buy glasses and contact lenses. She hires a survey company to research consumer preferences. A survey of 641 people provides the following information:
- 83 wear contact lenses.
- 442 wear glasses.
- 167 do not use corrective lenses.

What percent of Jennifer's customers might use a package deal? Use set notation in your answer.

5. Jacques is planning a winter ski holiday in the Canadian Rockies. Give four words or phrases that Jacques might use to search for information on the Internet. Use set theory to explain how quotation marks and the word "and" could help him refine his search.

6. A total of 58 teens attended a sports camp to train in at least one of three sports: swimming, cycling, and running.
   - 35 trained in swimming, 32 trained in cycling, and 38 trained in running.
   - 9 trained in swimming and cycling, but not in running.
   - 11 trained in cycling and running, but not in swimming.
   - 13 trained in swimming and running, but not in cycling.

   A triathlon consists of swimming, cycling, and running. How many teens might be training for the upcoming triathlon?



7. These nine attribute cards have three different shapes, numbers, and shadings (clear, striped, or solid).
   Determine three sets, with three cards in each set. Each set of three cards must have
   - the same number or three different numbers, and
   - the same shape or three different shapes, and
   - the same shading or three different shadings.

   All the cards can be used more than once.

8. Travis wants to buy a specific model of car. He goes into a car dealership in Medicine Hat, but the dealer does not have this model. The dealer searches the database and discovers that a Camrose dealership has four models, a Red Deer dealership has six models, a Sherwood Park dealership has five models, and a Lethbridge dealership has two models.
   a) What other attributes can the dealer use to narrow down the choices?
   b) How might the dealer prioritize the search?

9. John was asked to solve the following problem:
   240 students were surveyed to determine which restaurants they like.
   - 90 like Chicken and More.
   - 90 like Fast Pizza.
   - 90 like Gigantic Burger.
   - 37 like Chicken and More and Fast Pizza, but not Gigantic Burger.
   - 19 like Chicken and More and Gigantic Burger, but not Fast Pizza.
   - 11 like Fast Pizza and Gigantic Burger, but not Chicken and More.
   - 13 like all three restaurants.

   How many students do not like any of these restaurants?

   John solved the problem as follows:

   John's Solution:
   I added up the first six results of the survey and subtracted the number of students who ate at all three restaurants. Then I subtracted this value from the total number of students surveyed.

   $$90 + 90 + 90 + 37 + 19 + 11 - 13 = 324$$
   $$240 - 324 = -84$$

   This answer is not possible, so I knew that I made an error.

   What error did John make? What is the correct answer?

10. Wilson is searching online for information about local colleges and their athletics programs. He is interested in colleges in Edmonton or Calgary, but not universities.

a) His first search term is *colleges*. How can he categorize colleges in Edmonton or Calgary?

b) Since Wilson is interested in colleges and their athletics programs, should he use "and" or "or" to connect them?

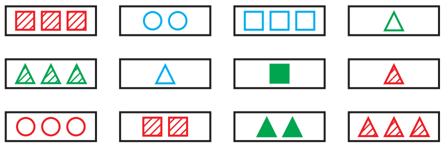
c) Should Wilson use "and" or "or" to search for one or the other city?

d) To exclude universities, Wilson used the notation –*university*. The minus sign means "not." What might Wilson's search instructions look like?

e) Try searching for the information that Wilson wants. What is the smallest number of hits you found?

f) Represent your results in a Venn diagram.

11. These 12 cards have three different colours, shapes, numbers, and shadings.



Determine six sets of cards, with three cards in each set. Each set of three cards must have

- the same number or three different numbers, and
- the same shape or three different shapes, and
- the same colour or three different colours, and
- the same shading or three different shadings.

All the cards can be used more than once.

12. The cards in question 11 are part of a complete deck of cards. Determine the following amounts:

a) $n(D)$, the total number of cards in the deck

b) $n(T)$, the total number of triangle cards in the deck

c) $n(G)$, the total number of green cards in the deck

d) $n(S)$, the total number of cards with shading

e) $n(T \cup G)$

f) $n(G \cap S)$

13. A small web-hosting service specializes in websites involving outdoor activities.

- 35 sites involve boats: 20 of these sites deal with fishing boats and 25 deal with power boats.
- 21 sites involve fishing: these sites include all the sites that deal with fishing boats; 3 sites deal with fly fishing.

a) How many sites from this service would appear in a search for *fishing boats*? Explain.

b) Why might a search for *fishing* and *boats* turn up a different result than a search for "*fishing boats*"?

c) If the only search word was *fishing*, how many results would not involve boats?

Closing

14. James searched for "*string bean*" on the Internet with quotation marks. Elinor searched for *string bean* without quotation marks. Did they get the same results? Explain, using set theory and a Venn diagram.

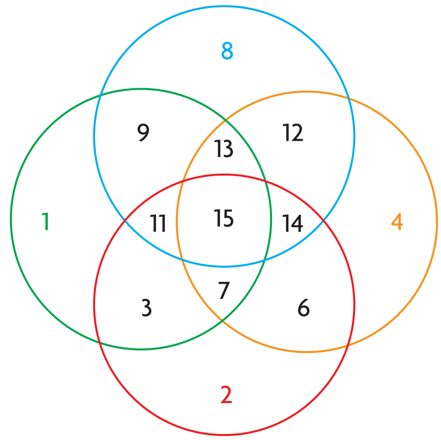
Extending

15. a) Four sets, A, B, C, and D, all intersect. Represent this using a Venn diagram. You do not need to limit the shapes to circles.

b) Number each area in your Venn diagram.

c) List the set combinations for each set. For example, the area that shows the intersection of only A, B, and C would be $(A \cap B \cap C) \setminus D$.

16. Explain why the following Venn diagram is not adequate to show four intersecting sets.



Math in Action

Relevant Hits

Have you noticed that many of the hits in an Internet search are not relevant to the topic you are searching?

Choose a topic that interests you. Decide on keywords to help you narrow your search.

Will your search involve the union of these words, the intersection of these words, or both? Are there any exact phrases you want to use, or do you just want hits that contain the words or variations of the words?

- Try searching with quotation marks, without quotation marks, and with the words "or" or "and." Adjust your search tools until you are satisfied with your results.
- Record the exact search you used, and explain why it worked. Show your results using a Venn diagram. For example, suppose that you want to adopt a dog. Your search might look like this:

  "golden retriever" or "Labrador retriever" and Alberta
- Does your browser's Advanced Search feature give you any more ideas on how to narrow your search?

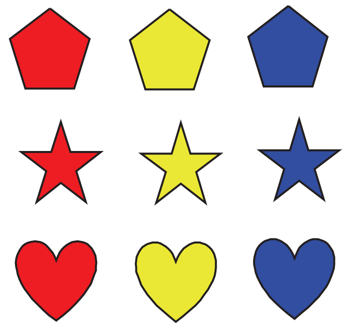
Applying Problem-Solving Strategies

Analyzing a Logic Puzzle

YOU WILL NEED

- coloured markers
- scissors
- Puzzle Shapes

Logic can often help you solve a puzzle.



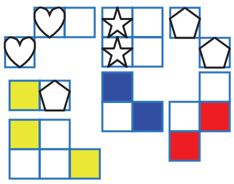
The Puzzle

A. In this activity, you will use all nine puzzle shapes and the given clues to form a three-by-three square. (You will need one of each puzzle shape in each colour.) The following hints will help you understand the clues:

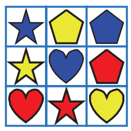
- A square with a solid colour means that the square must contain a shape of this colour.
- A hollow shape in a square means that the square must contain this shape.
- A solid symbol in a square means that square must contain this symbol.

Study Carol's solution to see how she used the clues she was given.

Carol's Clues

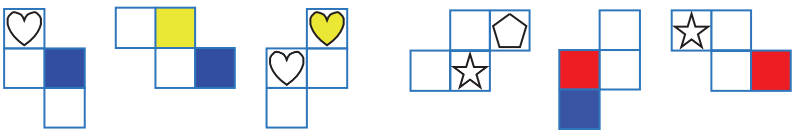


Carol's Solution



B. Use the nine puzzle shapes and the following clues to form a three-by-three square.

Your Clues



The Strategy

C. Describe the strategy you used to solve the puzzle in part B.

D. Create your own solution for a puzzle, and develop clues to solve the puzzle.

E. Give your clues to a partner. Can your partner solve your puzzle?

F. How could you adjust your clues to change the level of difficulty of your puzzle?