2.1 Counting Principles

YOU WILL NEED
- calculator
- standard deck of playing cards

EXPLORE...
- Suppose you roll a standard red die and a standard blue die at the same time. Describe the **sample space** for this experiment by listing all the different possible outcomes. How many different outcomes are there?

GOAL

Determine the Fundamental Counting Principle and use it to solve problems.

INVESTIGATE the Math

Serge lives in Winnipeg. This summer he plans a sight-seeing trip that includes visiting his family in Regina and Saskatoon. There are many places he might visit on the trip, but he knows he will stop in Regina to visit his brother and then in Saskatoon to visit his parents. He has chosen and mapped out three different routes he can take from Winnipeg to Regina and two different routes he can take from Regina to Saskatoon.

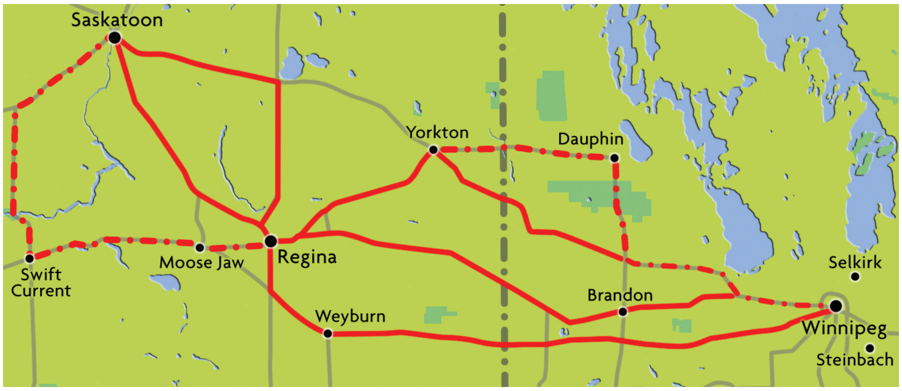
? How does the number of routes he has chosen between Winnipeg and Regina and between Regina and Saskatoon relate to the total number of routes he could take from Winnipeg to Saskatoon?

A. Sketch Serge's map and label the highlighted routes from Winnipeg to Regina and Regina to Saskatoon A to E.

B. Complete an **outcome table** to show the sample space for this situation. Count all the possible routes Serge can take from Winnipeg to Saskatoon.

C. Draw a **tree diagram** to visualize and count all the possible routes he can take from Winnipeg to Saskatoon, through Regina. Compare the results to your answer in part B. What do you notice?

D. Serge is thinking about adding a fourth route from Winnipeg to Regina, through Dauphin and Yorkton, and a third route from Regina to Saskatoon, through Moose Jaw and Swift Current. Make a **conjecture** about the number of routes he can now take from Winnipeg to Saskatoon via Regina.



E. Test your conjecture using an organized list, outcome table, or tree diagram.

F. Make a conjecture about how the number of routes between Winnipeg and Regina and between Regina and Saskatoon relates to the total number of routes he can take from Winnipeg to Saskatoon.

G. Let a represent the number of routes between Winnipeg and Regina and b represent the number of routes between Regina and Saskatoon. Create an expression that relates a and b to the number of possible routes he can take to go from Winnipeg to Saskatoon.

Reflecting

H. What type of reasoning did you use to arrive at your conjecture in part D? Explain how you know.

I. When figuring out the total number of routes Serge can take from Winnipeg to Saskatoon via Regina, which of the following applies?

- You need to consider the number of routes from Winnipeg to Regina AND the number of routes from Regina to Saskatoon.
- You need to consider the number of routes from Winnipeg to Regina OR the number of routes from Regina to Saskatoon.

What conclusion can you draw?

J. Discuss the advantages and disadvantages of the three different strategies (outcome table, tree diagram, and calculating) used to solve this counting problem.

APPLY the Math

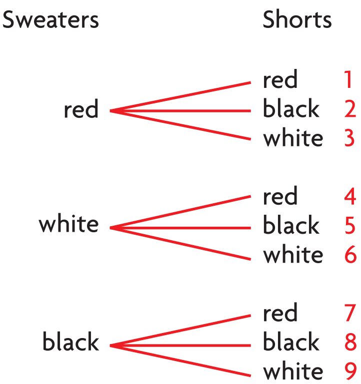
EXAMPLE 1 Selecting a strategy to solve a counting problem

Hannah plays on her school soccer team. The soccer uniform has:

- three different sweaters: red, white, and black, and
- three different shorts: red, white, and black.

How many different variations of the soccer uniform can the coach choose from for each game?

Hannah's Solution: Using a tree diagram



I created a tree diagram, considering the types of sweaters and shorts that are possible. Each branch of the diagram represents a different variation of the soccer uniform:

1. red sweater and red shorts
2. red sweater and black shorts
3. red sweater and white shorts
4. white sweater and red shorts
5. white sweater and black shorts
6. white sweater and white shorts
7. black sweater and red shorts
8. black sweater and black shorts
9. black sweater and white shorts

There are nine different variations of the soccer uniform to choose from.

I counted the number of branches in my tree diagram.

Mandy's Solution: Using the Fundamental Counting Principle

The number of uniform variations, U, is related to the number of sweaters and the number of shorts:

$$U = (\text{number of sweaters}) \cdot (\text{number of shorts})$$
$$U = 3 \cdot 3 \text{ or } 9$$

Since the coach can choose the type of sweater AND the type of shorts, I knew that I could use the **Fundamental Counting Principle** to calculate the number of uniform variations.

There are nine different variations of the soccer uniform to choose from.

I was able to determine the number of uniform variations without actually counting each one.

Your Turn

a) Next season, the coach is adding a fourth style for shorts: red/black stripes. Create an organized list, an outcome table, or a tree diagram to count all the different uniform variations.

b) Confirm your count using the Fundamental Counting Principle.

Fundamental Counting Principle

If there are *a* ways to perform one task and *b* ways to perform another, then there are *a* · *b* ways of performing both.

EXAMPLE 2 Solving a counting problem by extending the Fundamental Counting Principle

A luggage lock opens with the correct three-digit code. Each wheel rotates through the digits 0 to 9.

a) How many different three-digit codes are possible?
b) Suppose each digit can be used only once in a code. How many different codes are possible when repetition is not allowed?

Jake's Solution

a) The number of different codes, C, is related to the number of digits from which to select on each wheel of the lock, D:

To set the correct code, each wheel must be turned to display the correct digit. I reasoned that I could extend the Fundamental Counting Principle to more than two tasks, since I am selecting a number on wheel A AND selecting a number on wheel B AND selecting a number on wheel C.

There are 10 digits that can be selected on each wheel.

$$C = D_1 \cdot D_2 \cdot D_3$$
$$C = 10 \cdot 10 \cdot 10$$
$$C = 1000$$

There are 1000 different three-digit codes on this type of lock.

I was able to determine the number of codes using the Fundamental Counting Principle.

b) The number of different codes, N, is related to the number of digits from which to select on each wheel of the lock, W:

$$N = W_1 \cdot W_2 \cdot W_3$$
$$N = 10 \cdot 9 \cdot 8$$
$$N = 720$$

The problem is almost the same as the one in part a), except each digit can be used only once in each code. That means:

- for the first digit, you have 10 digits to select from, but
- the number of selections for wheel B decreases by 1 to account for the digit used for wheel A, and
- the number of selections for wheel C decreases by 2 to account for the digits used for wheels A and B.

There are only 720 different three-digit codes when the digits cannot repeat.

It makes sense that there are fewer codes without repetition, since there are restrictions on the digits that can be selected for wheels B and C.

Your Turn

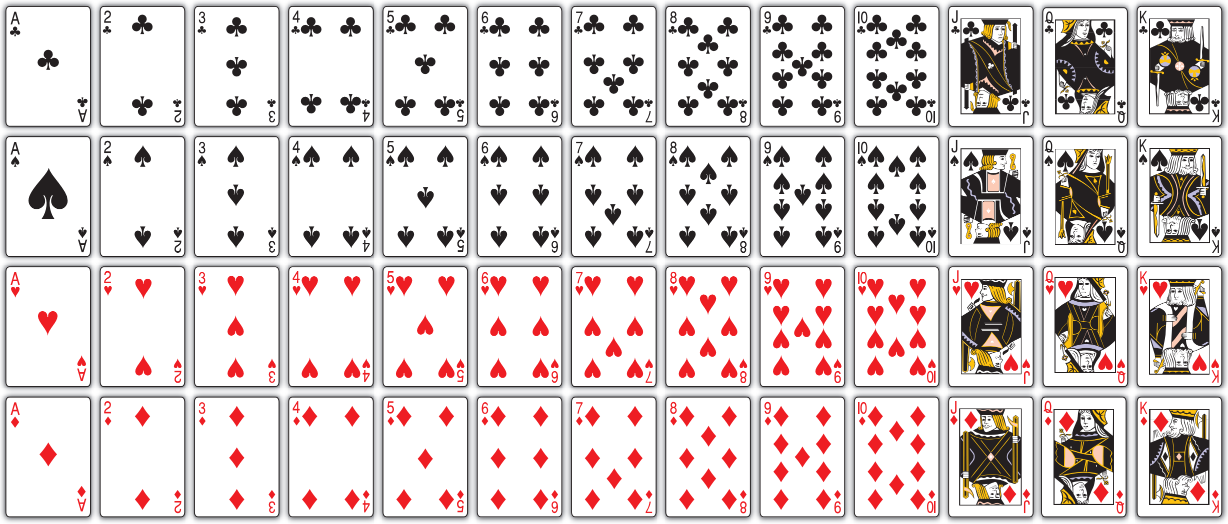
Suppose you buy a bicycle lock that opens using a five-digit code set by rotating five wheels through the digits 0 to 9. Which lock would be more secure?

- Lock A, which allows codes with repeating digits
- Lock B, which uses codes that do not allow repeating digits

Explain your answer.

EXAMPLE 3 Solving a counting problem when the Fundamental Counting Principle does not apply

A standard deck of cards contains 52 cards as shown.



There are four suits, two red and two black, with 13 cards in each suit.

Count the number of possibilities of drawing a single card and getting:

a) either a black face card or an ace

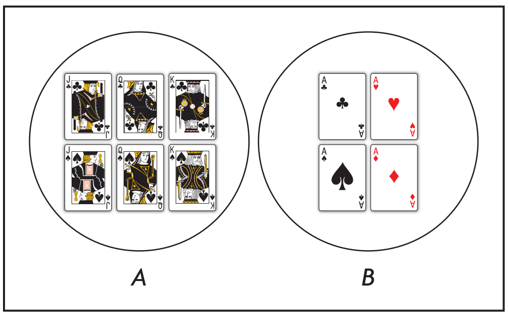
b) either a red card or a 10

Christian's Solution

a) Event A: Draw a black face card.
OR
Event B: Draw an ace.

I let A represent the set of six black face cards and B represent the set of four aces.

There are two events in this situation. I knew that I needed to determine the number of cards in the deck that are either black face cards OR aces.



I drew a Venn diagram to visualize how the two events relate. These events are mutually exclusive.

I knew that I needed to determine the number of elements in the union of two sets with no elements in common.

$$n(A \cup B) = n(A) + n(B)$$
$$n(A \cup B) = 6 + 4$$
$$n(A \cup B) = 10$$

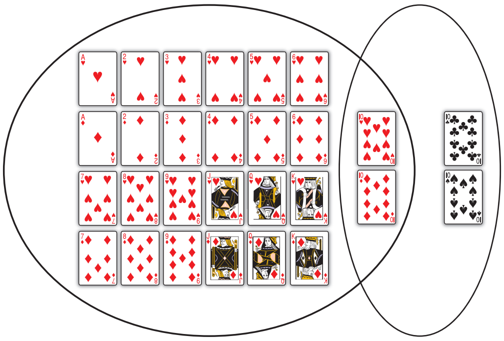
There are 10 ways to draw a single card and get either a black face card or an ace.

I knew I could add the number of ways that each event can occur. I knew I couldn't use multiplication (the Fundamental Counting Principle), since drawing a black face card AND drawing an ace is not possible on a single draw. I counted the number of black face cards and the number of aces in the deck and added.

b) Event C: Draw a red card.
OR
Event D: Draw a 10.

I let C represent the set of 26 red cards and I let D represent the set of four 10s.

There are two events in this situation. I knew that I needed to determine the number of cards in the deck that are either red cards OR 10s.



I drew a Venn diagram to visualize how the two events relate. These events are not mutually exclusive. I could see that I had to be careful of overlap, because there are two cards that can be both red and a 10, and I didn't want to count them twice.

I knew that I needed to determine the number of elements in the union of two sets that have common elements.

$$n(C \cup D) = n(C) + n(D) - n(C \cap D)$$
$$n(C \cup D) = 26 + 4 - 2$$
$$n(C \cup D) = 28$$

I knew I could add the number of ways that each event can occur but I had to subtract the number of elements in their intersection, the two red 10s. I added the number of red cards and the number of 10s and then subtracted the number of red 10s.

I used the Principle of Inclusion and Exclusion. I included the elements in the union of both sets and then excluded the elements in their intersection.

There are 28 ways to draw a single card and get either a red card or a 10.

Your Turn

Suppose you want to determine the number of possibilities of drawing a single card and getting:

- Event A: a club or a red card
- Event B: a heart or a 2

How would your strategies be the same? How would they be different?

In Summary

Key Ideas

- The Fundamental Counting Principle applies when tasks are related by the word AND.
- The Fundamental Counting Principle states that if one task can be performed in a ways and another task can be performed in b ways, then both tasks can be performed in $a \cdot b$ ways.

Need to Know

- The Fundamental Counting Principle can be extended to more than two tasks: if one task can be performed in a ways, another task can be performed in b ways, another task in c ways, and so on, then all these tasks can be performed in $a \cdot b \cdot c \ldots$ ways.

- The Fundamental Counting Principle does not apply when tasks are related by the word OR. In the case of an OR situation,
  - if the tasks are mutually exclusive, they involve two **disjoint** sets, A and B:

    $n(A \cup B) = n(A) + n(B)$
  - if the tasks are not mutually exclusive, they involve two sets that are not disjoint, C and D:

    $n(C \cup D) = n(C) + n(D) - n(C \cap D)$

  The Principle of Inclusion and Exclusion must be used to avoid counting elements in the intersection of the two sets more than once.

- Outcome tables, organized lists, and tree diagrams can also be used to solve counting problems. They have the added benefit of displaying all the possible outcomes, which can be useful in some problem situations. However, these strategies become difficult to use when there are many tasks involved and/or a large number of possibilities for each task.

CHECK Your Understanding

1. Cam is on vacation. In his suitcase he has three golf shirts (red, blue, and green) and two pairs of shorts (khaki and black).

a) Use an outcome table to count the total number of outfit variations he has for golfing.

b) Use the Fundamental Counting Principle to verify your result in part a).

2. Missy is buying her first new car. The model she wants comes in four colours (red, black, white, and silver) and she has a choice of leather or cloth upholstery.

a) Use a tree diagram to count all the upholstery–colour choices that are available.

b) Use the Fundamental Counting Principle to verify your result in part a).

3. For each situation below, indicate whether the Fundamental Counting Principle applies and explain how you know.

a) Counting the number of possibilities when rolling a 3 or a 6 with a standard die

b) Counting the number of outfit variations when selecting a shirt, a tie, and shoes to wear to the semiformal dance

c) Counting the number of possibilities when picking the winner in a stock car race in either the fourth, fifth, or sixth race of the evening

d) Counting the number of possibilities to choose from when buying a car with either standard or automatic transmission, air conditioning or not, power windows or not, and GPS navigation or not

PRACTISING

4. a) Kim plays hockey on the Lloydminster Ice Cats. Her team is in a best-two-out-of-three playoff series. Create a tree diagram to show all the win–loss possibilities for her team.

b) Use your tree diagram to count the number of ways Kim's team can win the series despite losing one game.

5. Xtreme clothing company makes snowboarding pants in five colours and sizes of small, medium, large, and extra large. How many different colour–size variations of snowboarding pants does this company make?

6. A computer store sells 5 different desktop computers, 4 different monitors, 6 different printers, and 3 different software packages. How many different computer systems can the employees build for their customers?

7. Jeb's Diner offers a lunch special. You have a choice of 3 soups, 5 sandwiches, 4 drinks, and 2 desserts. How many meals are possible if you choose one item from each category?

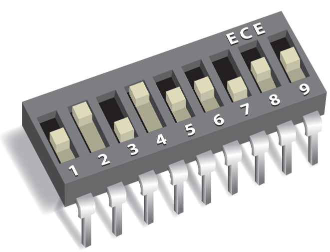
8. Tom likes rap music and classic rock. His friend Charlene has 8 rap CDs, 10 classic rock CDs, and 5 country and western CDs in her car. How many CDs can Charlene select from to play in her car stereo that will match Tom's musical tastes?

9. Rachelle's bank card has a five-digit PIN where each digit can be 1 to 9.
   a) How many PINs are possible if each digit can repeat?
   b) How many PINs are possible if each digit can be used only once?

10. Computers code information in a binary sequence, using 0 or 1 for each term in the sequence. Each sequence of eight terms is called a byte (for example, 00110010). How many different bytes can be created?

11. a) A country's postal code consists of six characters. The characters in the odd positions are upper-case letters, while the characters in the even positions are digits (0 to 9). How many postal codes are possible in this country?
    b) Canadian postal codes are similar, except the letters D, F, I, O, and U can never appear. (This is because they might be mistaken for the letters E or V or the numbers 0 or 1.) How many postal codes are possible in Canada?

12. A small town in Manitoba has a phone area code of 204 and two different three-digit prefixes as shown: 204-945-□□□□ or 204-940-□□□□. How many different phone numbers are possible for this town?



13. The code to a garage door opener is programmed by moving each of nine switches to any one of three positions. How many different codes are possible?

14. A vehicle rental company has 8 pickup trucks, 10 passenger vans, 35 cars, and 12 sports utility vehicles for rent. How many choices does a customer have when renting just 1 vehicle?

15. The "Pizza Shoppe" offers these choices for each pizza:
- thin or thick crust
- regular or whole-wheat crust
- 2 types of cheese
- 2 types of tomato sauce
- 20 different toppings

Determine the number of different pizzas that can be made as follows:
a) A pizza with any crust, cheese, tomato sauce, and 1 topping
b) A pizza with a thin whole-wheat crust, tomato sauce, cheese, and no toppings

16. An Alberta licence plate has three letters followed by three digits; for example, ABC 123. The letters I and O are not used to avoid confusion with the digits 1 and 0.
a) How many different Alberta licence plates are possible?
b) Because of the growing number of vehicles, the province is changing to plates with three letters followed by four digits; for example, ABC 1234. How many more licence plates are possible?

Closing

17. Counting problems often involve several tasks that are described using the words AND and OR. What is the mathematical meaning behind these words, and how does this affect the strategy you would use to solve counting problems that involve these words? Support your answer using relevant examples.

Extending

18. **a)** Determine the likelihood that each of the following events can occur using a standard deck of cards.
i) Drawing a king or a queen
ii) Drawing a diamond or a club
iii) Drawing an ace or a spade
b) Does the Fundamental Counting Principle apply to any situation in part a)? Explain.

19. How many two-digit numbers are not divisible by either 2 or 5?

20. A test has 10 true-false questions. A student attempts every question by guessing. What is the likelihood that the student will get a perfect score?

21. Recall Jeb's Diner and the lunch special from question 7. There are 3 soups, 5 sandwiches, 4 drinks, and 2 desserts to choose from. How many meals are possible if you do not have to choose an item from a category?