2.2 Introducing Permutations and Factorial Notation

YOU WILL NEED

- calculator

GOAL

Use factorial notation to solve simple permutation problems.

EXPLORE…

- How many different single-file lines are possible when Melissa, Nika, and François line up at a cashier at a fast-food restaurant?

LEARN ABOUT the Math

Naomi volunteers after school at a daycare centre in Whitehorse, Yukon. Each afternoon, around 4 p.m., she lines up her group of children at the fountain to get a drink of water.

permutation

An arrangement of distinguishable objects in a definite order. For example, the objects *a* and *b* have two permutations, *ab* and *ba*.

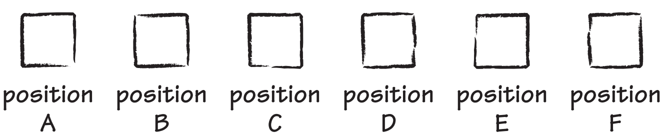
? How many different arrangements of children can Naomi create for the lineup for the water fountain if there are six children in her group?

EXAMPLE 1 Solving a counting problem where order matters

Determine the number of arrangements that six children can form while lining up to drink.

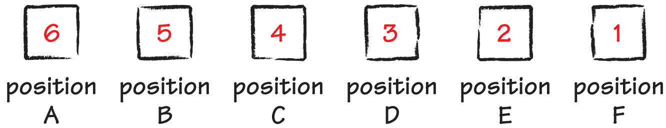
Evgeni's Solution

There are six children in the lineup, so there are six possible positions:



The problem involves arranging six children in a lineup. Each time you change the order of the children in the lineup, you get a different arrangement or **permutation** of the children.

I created a box diagram to help me visualize the problem.



I knew there were six possible children that could be in position A, since there are six children in the group. I then realized there would be only five children available for position B, since one of the six children would already be in position A. The number of possibilities for each position has to decrease by 1 to account for the children placed in previous positions.

Let L represent the total number of permutations:

$L = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 6!$

I wrote a product for the total number of permutations possible and then wrote it in a simpler form using **factorial notation**. I can now call the product 6 factorial.

$L = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 30 \cdot 4 \cdot 6$
$L = 120 \cdot 6$
$L = 720$

I used the product form to evaluate 6!. Because of the numbers involved, I was able to use mental math.

There are 720 permutations of the six children at the fountain.

Reflecting

A. Why did Evgeni multiply the number of possibilities for each position?

B. Why did the factors in Evgeni's product decrease by 1 each time?

C. Use n to represent the number of children in a group to be lined up. What expression represents the total number of permutations possible?

factorial notation

A concise representation of the product of consecutive descending natural numbers:
$n! = n(n - 1)(n - 2)...(3)(2)(1)$
For example:
$4! = 4 \cdot 3 \cdot 2 \cdot 1$

Communication *Tip*

Since factorial notation is defined only for natural numbers, expressions like $(-2)!$ and $\frac{1}{2}!$ have no meaning.

APPLY the Math

EXAMPLE 2 Evaluating numerical expressions involving factorial notation

Evaluate the following.

a) $10!$ **b)** $\frac{12!}{9!3!}$

Connie's Solution: Calculating by hand

a) $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$10! = 3\ 628\ 800$

I wrote 10! as a product. I decided to use a calculator to evaluate the product, since I couldn't figure out a mental math or pencil-and-paper strategy that would be practical.

b) $\frac{12!}{9!3!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)(3 \cdot 2 \cdot 1)}$

I wrote each factorial as a product.

$\frac{12!}{9!3!} = \frac{12}{3} \cdot \frac{11}{2} \cdot \frac{10}{1} \cdot \frac{9}{9} \cdot \frac{8}{8} \cdot \frac{7}{7} \cdot \frac{6}{6} \cdot \frac{5}{5} \cdot \frac{4}{4} \cdot \frac{3}{3} \cdot \frac{2}{2} \cdot \frac{1}{1}$

$\frac{12!}{9!3!} = \frac{12}{3} \cdot \frac{11}{2} \cdot 10 \cdot \frac{9!}{9!}$

I simplified the expression by aligning common factors in the numerator and denominator. Then I realized I could write part of the expression as $\frac{9!}{9!}$, which is equal to 1.

$\frac{12!}{9!3!} = \frac{12}{3} \cdot \frac{11}{2} \cdot 10 \cdot 1$

I simplified further by dividing 12 by 6, the product of the denominators 2×3.

$\frac{12!}{9!3!} = 2 \cdot 11 \cdot 10$

$\frac{12!}{9!3!} = 22 \cdot 10$

Since 11 multiplied by 2 is 22, I used mental math to determine the product of 22 and 10.

$\frac{12!}{9!3!} = 220$

Your Turn

Create a permutation problem that can be solved by evaluating $10!$.

EXAMPLE 3 Simplifying an algebraic expression involving factorial notation

Simplify, where $n \in \mathrm{N}$.

a) $(n + 3)(n + 2)!$ **b)** $\dfrac{(n + 1)!}{(n - 1)!}$

Phil's Solution

a) $(n + 3)(n + 2)! = (n + 3)[(n + 2)(n + 1)(n)(n - 1)...(3)(2)(1)]$

The expression indicates to multiply $(n + 3)$ by the factors of $(n + 2)!$, so I wrote $(n + 2)!$ as a product.

$(n + 3)(n + 2)! = (n + 3)(n + 2)(n + 1)(n)(n - 1)...(3)(2)(1)$
$(n + 3)(n + 2)! = (n + 3)!$

I know that $5! = 5(4!)$, so it makes sense that $n! = n[(n - 1)!]$. I used this reasoning to simplify the product, since $(n + 3)$ and the factors of $(n + 2)!$ are all the factors of $(n + 3)!$.

b) $\dfrac{(n + 1)!}{(n - 1)!} = \dfrac{(n + 1)(n)(n - 1)(n - 2)(n - 3)...(3)(2)(1)}{(n - 1)(n - 2)(n - 3)...(3)(2)(1)}$

I wrote both $(n + 1)!$ and $(n - 1)!$ as products.

Then I looked for common factors in the numerator and denominator.

$\dfrac{(n + 1)!}{(n - 1)!} = \dfrac{(n + 1)(n)(n - 1)!}{(n - 1)!}$

$\dfrac{(n + 1)!}{(n - 1)!} = (n + 1)(n)$

$\dfrac{(n + 1)!}{(n - 1)!} = n^2 + n$

I realized that I could write the expressions in both the numerator and denominator using the common factor $(n - 1)!$.

I simplified the expression using the fact that $\dfrac{(n - 1)!}{(n - 1)!} = 1$

Your Turn

When $(n + 3)(n + 2)!$ is multiplied by $(n + 4)$ and then divided by $(n + 2)!$, what is the result?

EXAMPLE 4 Solving an equation involving factorial notation

Solve $\frac{n!}{(n-2)!} = 90$, where $n \in I$.

Katrina's Solution

$$\frac{n(n-1)(n-2)(n-3)\ldots(3)(2)(1)}{(n-2)(n-3)\ldots(3)(2)(1)} = 90$$

I wrote each factorial as a product.

Then I looked for common factors in the numerator and denominator.

$$\frac{n(n-1)(n-2)!}{(n-2)!} = 90$$

I realized that I could write the expressions in the numerator and denominator using the common factor of $(n-2)!$.

$$n(n-1) = 90$$
$$n^2 - n = 90$$

I simplified the expression using the fact that $\frac{(n-2)!}{(n-2)!} = 1$

$$n^2 - n - 90 = 0$$
$$(n-10)(n+9) = 0$$
$$n - 10 = 0 \quad \text{or} \quad n + 9 = 0$$
$$n = 10 \qquad n = -9$$

Since the result was a quadratic equation, I rearranged the equation so that the right side was zero. I factored the left side and used each factor to solve the equation.

Check $n = 10$.

LS	RS
$\frac{10!}{(10-2)!}$	90
$\frac{10!}{8!}$	
$\frac{10 \cdot 9 \cdot 8!}{8!}$	
$10 \cdot 9$	
90	

Check $n = -9$.

LS	RS
$\frac{(-9)!}{(-9-2)!}$	90
$\frac{(-9)!}{(-11)!}$	
is undefined	

I verified my solutions by substituting into the original equation.

Although -9 is an integer, since factorial notation is defined only for natural numbers, $n = -9$ is undefined and therefore is extraneous.

There is one solution, $n = 10$.

Only $n = 10$ results in the left side equalling the right side, and $10 \in I$.

Your Turn

Solve $\frac{(n+4)!}{(n+2)!} = 6$, where $n \in I$.

In Summary

Key Ideas

- A permutation is an arrangement of objects in a definite order, where each object appears only once in each arrangement. For example, the set of three objects *a*, *b*, and *c* can be listed in six different ordered arrangements or permutations:

	Position 1	Position 2	Position 3
Permutation 1	*a*	*b*	*c*
Permutation 2	*a*	*c*	*b*
Permutation 3	*b*	*a*	*c*
Permutation 4	*b*	*c*	*a*
Permutation 5	*c*	*a*	*b*
Permutation 6	*c*	*b*	*a*

- The expression $n!$ is called *n* factorial and represents the number of permutations of a set of *n* different objects and is calculated as

$$n! = n(n - 1)(n - 2)...(3)(2)(1)$$

Need to Know

- In the expression $n!$, the variable *n* is defined only for values that belong to the set of natural numbers; that is, $n \in \{1, 2, 3,...\}$.

CHECK Your Understanding

1. Evaluate the following expressions.

a) $6!$ **c)** $\frac{5!}{3!}$ **e)** $3! \cdot 2!$

b) $9 \cdot 8!$ **d)** $\frac{8!}{7!}$ **f)** $\frac{9!}{4! \cdot 3!}$

2. a) How many permutations are possible of Ken, Sarah, and Raj when they line up to buy a slice of pizza? Describe your strategy.

b) Express the number of permutations using factorial notation.

3. Write the following expressions using factorial notation.

a) $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ **c)** $\frac{15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1}$

b) $9 \cdot 8 \cdot 7$ **d)** $100 \cdot 99$

4. Which expressions are undefined? Explain how you know.

a) $(-4)!$ **b)** $7!$ **c)** $5.5!$ **d)** $\frac{3}{4}!$

PRACTISING

5. Evaluate the following expressions.

a) $8 \cdot 7 \cdot 6!$

b) $\dfrac{12!}{10!}$

c) $\dfrac{8!}{2! \cdot 6!}$

d) $\dfrac{7 \cdot 6!}{5!}$

e) $4\left(\dfrac{6!}{2! \cdot 2!}\right)$

f) $4! + 3! + 2! + 1!$

6. Simplify each of the following expressions, where $n \in I$.

a) $\dfrac{n!}{(n - 1)!}$

b) $(n + 4)(n + 3)(n + 2)!$

c) $\dfrac{(n + 1)!}{n!}$

d) $\dfrac{n!}{(n - 3)!}$

e) $\dfrac{(n + 5)!}{(n + 3)!}$

f) $\dfrac{(n - 2)!}{(n - 1)!}$

The Calgary Stampede, held every July, celebrates Western heritage. The first Stampede was held in 1912.

7. How many different permutations can be created when nine students line up to buy tickets for the afternoon rodeo at the Calgary Stampede?

8. The environmental club has five members. They want to select a president, a vice-president, a secretary, a treasurer, and a spokesperson. How many different ways can this be done?

9. Amy and Traci are planning a summer trip to Vancouver Island in British Columbia. They plan to spend six days in Tofino. While they are there, they have decided to take part in the following activities: whale watching, hiking, surfing, sea kayaking, snorkeling, and fishing. They plan to do a different activity each day. In how many different ways can they sequence these activities over the six days?

10. Visitors to a movie website will be asked to rank 28 movies. The website will present the movies in a different order for each visitor to reduce bias in the poll. How many permutations of the movie list are possible?

11. Solve for n, where $n \in I$.

a) $\dfrac{(n + 1)!}{n!} = 10$

b) $\dfrac{(n + 2)!}{n!} = 6$

c) $\dfrac{(n - 1)!}{(n - 2)!} = 8$

d) $\dfrac{3(n + 1)!}{(n - 1)!} = 126$

12. A baseball coach is determining the batting order for the nine players he is fielding. The coach has already decided that he wants the pitcher to hit in last position. How many different batting orders are possible?

13. On an assembly line for a company that makes digital cameras, seven-digit serial numbers are assigned to each camera under the following conditions:

- Only the digits from 3 to 9 are used.
- Each digit is to be used only once in each serial number.

How many different serial numbers are possible? Express your answer using factorial notation and explain why your answer makes sense.

14. A model train has an engine, a caboose, a tank car, a flat car, a boxcar, a livestock car, and a refrigerator car. How many different ways can the cars be arranged between the engine and the caboose?

15. Eight drivers have made it to the final chuckwagon race at Back to Batoche Days, although Brant's wagon is considered certain to win. In how many different orders can the eight chuckwagons finish, if Brant's wagon wins?

Each year for three days in July, Saskatchewan's Métis Nation gathers at the site of the Battle of Batoche for a festival called Back to Batoche Days. The Métis settled this area, northeast of Saskatoon, Saskatchewan, in the early 1870s.

Closing

16. Consider the word YUKON and all the ways you can arrange its letters using each letter only once.

a) One possible permutation is KYNOU. Write three other possible permutations.

b) Use factorial notation to represent the total number of permutations possible. Explain why your expression makes sense.

Extending

17. a) For what values of n is $n!$ greater than 2^n?

b) For what values of n is $n!$ less than 2^n?

18. Darlene and Arnold belong to the Asham Stompers, a 10-member dance troupe based in Winnipeg, Manitoba, that performs traditional Métis dances. During the Red River Jig, they always arrange themselves in a line, with Darlene and Arnold next to each other. How many different arrangements of the dancers are possible for the Red River Jig?