2.3 Permutations When All Objects Are Distinguishable

YOU WILL NEED

- calculator
- standard deck of playing cards

GOAL

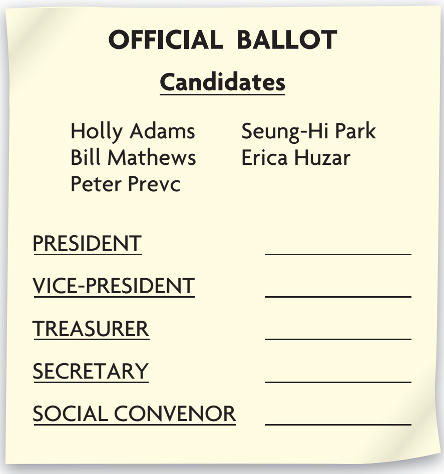
Determine the number of permutations of n objects taken r at a time, where $0 \leq r \leq n$.

EXPLORE...

- How many three-letter permutations can you make with the letters in the word MATH?

INVESTIGATE the Math

Chloe's school is holding elections for five positions on next year's student council.



? How can Chloe determine the number of ways she can fill out her ballot, if she considers voting for only some of these positions?

A. How many different ways can she fill out the ballot if she votes for all five positions? Write your answer using factorial notation.

B. Use the Fundamental Counting Principle to write the number of ways she can fill out her ballot for each of the following numbers of positions. Leave your answers in factored form.

i) She votes for only the president, vice-president, treasurer, and secretary.

ii) She votes for only the president, vice-president, and treasurer.

iii) She votes for only the president and vice-president.

iv) She votes for only the president.

C. Natalie claims that the expression $\frac{5!}{1!}$ represents the number of ways Chloe can fill out her ballot to vote for only the president, vice-president, treasurer, and secretary. Do you agree? Explain.

D. Use Natalie's idea to write expressions involving the quotient of two factorials for each of the other voting choices in part B.

E. How do the numbers in each expression in part D relate to the number of positions on the ballot and the number of positions she votes for?

F. Suppose there are n positions on the student council and Chloe votes for r of these positions, where $r \leq n$. Write an expression that represents the number of ways she could fill out her ballot.

Reflecting

G. Suppose there are 10 student council positions and she votes for only the president and vice-president. Use the expression you wrote in part F to determine the number of ways Chloe could fill out her ballot. Verify your answer using the Fundamental Counting Principle.

H. Recall your answer to part A. Does the expression you wrote in part F work for the situation described in part A? Explain.

I. What voting situation would the expression $\frac{10!}{(10-6)!}$ represent?

Communication | *Notation*

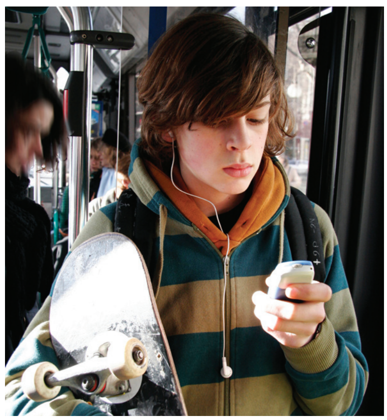
${}_nP_r$ is the notation commonly used to represent the number of permutations that can be made from a set of n different objects where only r of them are used in each arrangement, and $0 \leq r \leq n$.

When all available objects are used in each arrangement, n and r are equal, so the notation ${}_nP_n$ is used.

APPLY the Math

EXAMPLE 1 Solving a permutation problem where only some of the objects are used in each arrangement

Matt has downloaded 10 new songs from an online music store. He wants to create a playlist using 6 of these songs arranged in any order. How many different 6-song playlists can be created from his new downloaded songs?



Matt's Solution

$$_nP_r = \frac{n!}{(n-r)!}$$

When $n = 10$ and $r = 6$,

$$_{10}P_6 = \frac{10!}{(10-6)!}$$

$$_{10}P_6 = \frac{10!}{4!}$$

I needed to determine the number of different ordered arrangements that can be made from 10 songs, where each arrangement has only 6 songs.

I knew this was the same as determining the number of permutations possible for 10 different objects (n), where each permutation consists of only 6 of those objects (r).

$$_{10}P_6 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4!}$$

$$_{10}P_6 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$$

$$_{10}P_6 = 151\ 200$$

I simplified the expression using the fact that $\frac{4!}{4!} = 1$. Then I multiplied the remaining numbers using a calculator.

A B C D E F
G H I J A B
D A H F E I
B D J H A E

I checked to see if my answer was reasonable. I began by representing each of the 10 downloaded songs with a letter from A to J and then wrote some possible arrangements using 6 of these letters to represent each playlist.

Choices for each song in the playlist:

10	9	8	7	6	5
song 1	song 2	song 3	song 4	song 5	song 6

I then realized that I could also use the Fundamental Counting Principle. For the first song there are 10 choices, for the second song there are 9 left to choose from, for the third song, only 8 choices are left, and so on. I drew a box diagram to help me visualize this.

Number of different 6-song playlists:
$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = {}_{10}P_6$

There are 151 200 different 6-song playlists that can be created from 10 songs.

I could have used the $_nP_r$ formula or the Fundamental Counting Principle to solve this problem.

Your Turn

Determine all the possible 7-song playlists, then 8-song playlists, and finally 9-song playlists that you can create from 10 songs. How does the value of $_nP_r$ change as r gets closer to n? Is this what you would have predicted? Explain.

EXAMPLE 2 Defining 0!

Use the formula for ${}_nP_r$ to show that $0! = 1$.

Kat's Solution

$${}_nP_r = \frac{n!}{(n-r)!}$$

I wrote the formula for ${}_nP_r$.

$${}_nP_n = \frac{n!}{(n-n)!}$$

$${}_nP_n = \frac{n!}{0!}$$

I let $n = r$ and performed the substitution so I'd get an expression that contains 0! in the denominator.

I simplified.

$${}_nP_n = n(n-1)(n-2)...(3)(2)(1)$$

I knew that the notation ${}_nP_n$ represented the number of permutations that are possible from a group of n different objects using all the objects in each arrangement.

$${}_nP_n = n!$$

I knew by the Fundamental Counting Principle that $n! = n(n-1)(n-2)...(3)(2)(1)$, so ${}_nP_n$ must also be equal to $n!$.

Since,

$${}_nP_n = \frac{n!}{0!} \text{ and } {}_nP_n = n!$$

Since both expressions represent ${}_nP_n$, they must be equal to each other by the transitive property.

Then,

$$\frac{n!}{0!} = n!$$

$$0! = 1$$

I reasoned that the only way for $\frac{n!}{0!} = n!$ to be true was for 0! to be equal to 1.

Your Turn

State the values of n for which each expression is defined, where $n \in \mathrm{I}$.

a) $(n + 3)!$ **b)** $\frac{n!}{(n + 2)!}$

EXAMPLE 3 Solving a permutation problem involving cases

Tania needs to create a password for a social networking website she registered with. The password can use any digits from 0 to 9 and/or any letters of the alphabet. The password is case sensitive, so she can use both lower- and upper-case letters. A password must be at least 5 characters to a maximum of 7 characters, and each character can be used only once in the password. How many different passwords are possible?

Tania's Solution

Number of characters $= 10 + 2(26)$

Number of characters $= 62$

$${}_nP_r = \frac{n!}{(n-r)!}$$

I could use any of 10 digits and 52 letters to create a password. Any change in the order of the characters results in a different password, even if the characters are the same.

Case 1: 5-character passwords for $n = 62$ and $r = 5$

Case 2: 6-character passwords for $n = 62$ and $r = 6$

Case 3: 7-character passwords for $n = 62$ and $r = 7$

Since each character can be used only once, I knew that I needed to calculate the number of permutations possible for 62 objects, where each arrangement uses 5 OR 6 OR 7 of those objects.
That meant there were three cases to consider. I knew I could use the ${}_nP_r$ formula for each case.

Total number of passwords = number of five-character passwords + number of six-character passwords + number of seven-character passwords

So, I needed to determine the sum of all three cases to determine the total, because a password can consist of 5 OR 6 OR 7 characters.

Total number of passwords $= {}_{62}P_5 + {}_{62}P_6 + {}_{62}P_7$

Total number of passwords = 2 523 690 780 000

Since n is large, I used a calculator to enter each case and determine the sum.

There are 2 523 690 780 000 different 5-, 6-, and 7-character passwords using the 62 characters.

Your Turn

Suppose an online magazine requires each subscriber to have a password with exactly 8 characters, using the same requirements for characters described in the problem above. Which is more secure: the social networking website in Example 3 above or the magazine website? Explain.

EXAMPLE 4 Solving a permutation problem with conditions

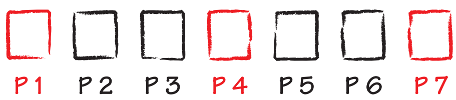
At a used car lot, seven different car models are to be parked close to the street for easy viewing.

a) The three red cars must be parked so that there is a red car at each end and the third red car is exactly in the middle. How many ways can the seven cars be parked?

b) The three red cars must be parked side by side. How many ways can the seven cars be parked?

Nick's Solution

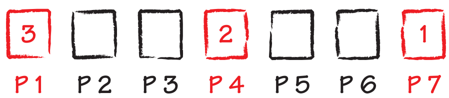
a) My diagram shows three red cars in positions P1, P4, and P7:



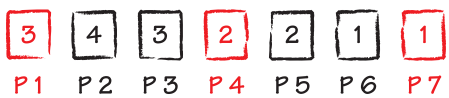
I created a box diagram to visualize the position of each car in the parking lot.

I knew that changing the order of the cars creates a different arrangement.

In this case, the cars in positions P1, P4, and P7 must be red.



I thought about the condition first. Since there are three red cars, I had three ways to place a red car in position P1, two ways for position P4, and one way for position P7.



That left placing the four non-red cars in the remaining positions using the same reasoning: four ways for position P2, three ways for position P3, two ways for position P5, and one way for position P6.

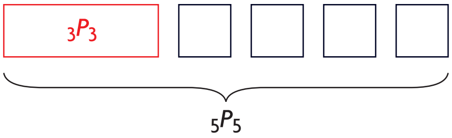
Let A represent the number of arrangements:

$$A = {}_3P_3 \cdot {}_4P_4$$
$$A = 3! \cdot 4!$$
$$A = 6 \cdot 24$$
$$A = 144$$

The cars can be parked 144 different ways.

From the diagram, I could see that I needed to calculate the number of permutations of the three red cars and the number of permutations of the four non-red cars.

I knew I could use the Fundamental Counting Principle to determine the product of the permutations of red and non-red cars, since the red cars must be positioned as shown AND the non-red cars must be positioned as shown to create a line of seven cars.

b) 

I used one red box to represent the three red cars because they had to be next to each other, and four boxes for the other four non-red cars.

I knew the three red cars could be arranged 3! ways within the red box. Then I thought of the block of red cars as one car in a five-car group.

Let B represent the number of arrangements:

$B = {}_3P_3 \cdot {}_5P_5$

$B = 3! \cdot 5!$

$B = 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$B = 720$

The cars can be parked 720 different ways.

I knew I could use the Fundamental Counting Principle to multiply the two conditions together because I am arranging the red cars within the red box AND arranging the five boxes (the red box and the four black boxes) to determine the total number of arrangements.

Your Turn

How many ways can the seven cars be parked if the three red cars must be parked side by side and the other four cars must also be side by side?

EXAMPLE 5 Comparing arrangements created with and without repetition

A social insurance number (SIN) in Canada consists of a nine-digit number that uses the digits 0 to 9. If there are no restrictions on the digits selected for each position in the number, how many SINs can be created if each digit can be repeated? How does this compare with the number of SINs that can be created if no repetition is allowed?

Caroline's Solution

Let S represent the number of social insurance numbers if digits are allowed to repeat:

$S = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

I began by determining how many nine-digit arrangements could be made if digits are allowed to repeat.

There are 10 digits from 0 to 9, and I assumed that for each position in the SIN there are 10 possible digits.

$S = 10^9$

$S = 1\ 000\ 000\ 000$

I noticed that the product formed a power with the number of digits to choose from as the base (n) and the number of those digits actually being used as the exponent (r).

$${}_nP_r = \frac{n!}{(n-r)!}$$

where $n = 10$ and $r = 9$

$${}_{10}P_9 = \frac{10!}{(10-9)!}$$

$${}_{10}P_9 = \frac{10!}{1!}$$

$${}_{10}P_9 = 10!$$

$${}_{10}P_9 = 3\ 628\ 800$$

Then I determined how many nine-digit arrangements could be made if digits are not allowed to repeat.

If each digit can be used only once, then each digit in the arrangement is different. This is the same as determining the number of permutations of 10 objects using 9 in each arrangement.

$1\ 000\ 000\ 000 - 3\ 628\ 800 = 996\ 371\ 200$

Therefore, 996 371 200 additional SINs can be made if repetition of digits is allowed.

The difference in the two situations represents the additional SINs that can be made if repetition is allowed.

Your Turn

a) In reality, the Canadian government does not use 0, 8, or 9 as the first digit when assigning SINs to citizens and permanent residents, and repetition of digits is allowed. How many nine-digit SINs do not start with 0, 8, or 9?

b) SINs starting with the digit 9 are issued to temporary residents. How many SINs are there in total?

In Summary

Key Ideas

- The number of permutations from a set of n different objects, where r of them are used in each arrangement, can be calculated using the formula

$$_nP_r = \frac{n!}{(n-r)!}, \text{ where } 0 \le r \le n$$

For example, if you have a set of three objects, a, b, and c, but you use only two of them at a time in each permutation, the number of permutations is

$$_3P_2 = \frac{3!}{(3-2)!} \text{ or } 6$$

	Position 1	Position 2
Permutation 1	*a*	*b*
Permutation 2	*a*	*c*
Permutation 3	*b*	*a*
Permutation 4	*b*	*c*
Permutation 5	*c*	*a*
Permutation 6	*c*	*b*

- When all n objects are used in each arrangement, n is equal to r and the number of arrangements is represented by $_nP_n = n!$.
- The number of permutations that can be created from a set of n objects, using r objects in each arrangement, where repetition is allowed and $r \le n$, is n^r. For example, the number of four-character passwords using only the 26 lower-case letters, where letters can repeat, is $26 \cdot 26 \cdot 26 \cdot 26 = 26^4$.

Need to Know

- If order matters in a counting problem, then the problem involves permutations. To determine all possible permutations, use the formula for $_nP_n$ or $_nP_r$, depending on whether all or some of the objects are used in each arrangement. Both of these formulas are based on the Fundamental Counting Principle.
- By definition,

$$0! = 1$$

As a result, any algebraic expression that involves factorials is defined as long as the expression is greater than or equal to zero. For example, $(n + 4)!$ is only defined for $n \ge -4$ and $n \in \mathrm{I}$.

- If a counting problem has one or more conditions that must be met,
  - consider each case that each condition creates first, as you develop your solution, and
  - add the number of ways each case can occur to determine the total number of outcomes.

CHECK Your Understanding

1. Evaluate the following expressions.
   a) ${}_5P_2$ c) ${}_{10}P_5$ e) ${}_7P_7$
   b) ${}_8P_6$ d) ${}_9P_0$ f) ${}_{15}P_5$

2. Katrina, Jess, Nazir, and Mohamad are running for student council.
   a) Create a list or use a tree diagram to count all the ways that a president and vice-president can be elected.
   b) Verify your results from part a) using the formula for ${}_nP_r$.

3. How many different ways can six different chocolate bars be distributed to each set of children, if each child is to receive only one?
   a) 4 children b) 1 child

4. Without calculating, predict which value is larger: ${}_{10}P_8$ or ${}_{10}P_2$. Explain your prediction.

PRACTISING

5. Nine girls are available to fill three positions on a hockey team: centre, winger, and defence. If all the girls are equally competent in each position, how many different ways can the positions be filled?

6. The photo club must select an executive committee to fill the roles of president, vice-president, treasurer, and secretary. All 15 members of the club are eligible. How many different executive committees are possible?

7. The MV UMIAVUT (umiavut means "our boat" in Inuktitut) is a cargo ship that sails the Northwest Passage. If it carries eight different coloured signal flags, how many different signals could be created by hoisting all eight flags on the ship's flagpole in different orders?

8. A wheelchair basketball team sold 5000 tickets in a draw to raise money. How many ways can the tickets be drawn to award the first, second, and third prizes if each winning ticket drawn is not put back prior to the next draw?

9. If a Canadian social insurance number begins with the digit 6, it indicates that the number was registered in Manitoba, Saskatchewan, Alberta, Northwest Territories, or Nunavut. If the number begins with a 7, the number was registered in British Columbia or the Yukon. How many different SINs can be registered in each of these groups of provinces and territories?

10. Sandy belongs to the varsity basketball team. There are 12 students on the team. How many ways can the coach select each of the following?

a) The starting five players (a point guard, a shooting guard, a small forward, a power forward, and a centre)

b) The starting five players, if the tallest student must start at the centre position

c) The starting five players, if Sandy and Natasha must play the two guard positions

11. For each expression, state the values of n for which the expression is defined.

a) $\frac{n!}{(n-1)!}$

b) $(n+4)(n+3)(n+2)!$

c) $\frac{(n+1)!}{n!}$

d) $\frac{(n+5)!}{(n+3)!}$

12. There are six different marbles in a bag. Suppose you reach in and draw one at a time, and do this four times. How many ways can you draw the four marbles under each of the following conditions?

a) You do not replace the marble each time.

b) You replace the marble each time.

c) Compare your answers for parts a) and b). Does it make sense that they differ? Explain.

13. How many ways can five different graduation scholarships be awarded to 20 students under each of these conditions?

a) No student may receive more than one scholarship.

b) There is no limit to the number of scholarships awarded to each student.

14. All phone numbers consist of a three-digit area code, a three-digit exchange, and then a four-digit number. A town uses the 587 area code and the exchange 355; for example, 587-355-1234. How many different phone numbers are possible in this town under the following conditions?

a) There are four different digits in the last four digits of the phone number.

b) At least one digit repeats in the last four digits of the phone number.

15. Solve each equation for n. State any restrictions on n.

a) ${}_{n}P_{2} = 20$ **b)** ${}_{n+1}P_{2} = 72$

16. Solve each equation for r. State any restrictions on r.

a) ${}_{6}P_{r} = 30$ **b)** $2({}_{7}P_{r}) = 420$

17. Show that for all values of n, where $n \in \mathrm{N}$,

$${}_{n}P_{n} = {}_{n}P_{n-1}$$

Closing

18. a) Describe how the formulas for calculating permutations, ${}_nP_n$ and ${}_nP_r$ where $r \leq n$, are the same and how they are different.

b) Provide an example of a counting problem that could be solved using each formula.

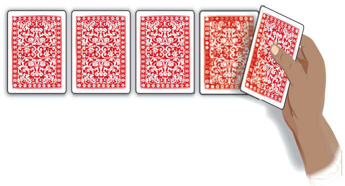
Extending

19. Five cards are drawn from a standard deck of cards and arranged in a row from left to right in the order in which they are drawn. Determine each of the following.

a) The number of possible arrangements

b) The likelihood that an arrangement contains black cards only

c) The likelihood that an arrangement contains hearts only



20. Show that

$$_{n+1}P_2 - {}_nP_1 = n^2$$

21. Show that for natural numbers n and whole numbers r, where $n > r$,

$$_nP_{r+1} = (n - r)_nP_r$$

Math in Action

Birthday Permutations

The days of the year (not including leap years) can be labelled 1 to 365. You can create a four-number sequence in ascending order to represent the birthdays of four people. For example, 3, 45, 230, 310 represents the birthdays of four people born on the 3rd, 45th, 230th, and 310th days of the year.

a) Create groups of four. Record the birthdates of everyone in your group as an ascending sequence of numbers from 1 to 365.

b) Use what you know about permutations to calculate the number of different birthday sequences that are possible for any four-person group.

c) Use your answer from part b) and what you know about permutations to calculate
- the percent of four-person birthday sequences that have four different birthdays
- the percent of four-person birthday sequences that have two or more birthdays on the same day

d) Use actual data collected from the groups in your classroom to calculate
- the percent of four-person birthday sequences with four different birthdays
- the percent of four-person birthday sequences with two or more birthdays on the same day

e) How do your calculations for part c) compare to your findings for part d)?